Volume 2, Issue 5, May - 2024 **ISSN(E):** 2938-3773

Web of Discoveries: Journal of Analysis and Inventions
webofjournals.com/index.php/3

# THE PROBLEM OF ACCELERATING THE ITERATION PROCESS IN SOLUTION OF THE DIRAC EQUATION

Abdunazarov Rabimkul
Senior Lecturer, Department of Applied Mathematics
Jizzakh Branch of the National University of Uzbekistan
rabimkulabdunazarov@gmail.com

Sharipova Sadoqat
Senior Lecturer, Department of Applied Mathematics
Jizzakh Branch of the National University of Uzbekistan
sadoqat_sh@jbnuu.uz

Kudaikulov Anarbay Kudaikulovich
Doctor of Physical and Mathematical Sciences, Academician Institute of Mechanics and Engineering Science Committee of Science Ministry of Science and Higher Education, Almaty

**Abstract**
This paper proposes an accelerated algorithm for solving the Dirac equation. The advantage of this method is that, thanks to the successful organization of the iterative process, a significant part of the found approximate solution is expressed through an elementary function.

**Keywords**: Dirac equations, eigenvalues, eigenfunctions, Dirac systems, successive approximation method.

## Introduction

Consider the Dirac equation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y' + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} Y = \lambda Y \qquad Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \tag{1}$$

with initial conditions

$$y_1(0) = cos\alpha, \qquad y_2(0) = -sin\alpha, \tag{2}$$

here $p(x)$, $q(x)$ are continuous functions in an arbitrary finite interval, $\lambda$ is a complex number. Problems (1)-(2) have been studied quite well and a review on this topic can be found in [1].

This paper proposes an accelerated algorithm for solving problem (1)-(2), which, thanks to a successful replacement of variables, the iterative process quickly converges to a solution, described by an elementary function, with a given accuracy. This approach to solving similar problems was practiced in the author's works [2]-[7].

The relevance of this problem formulation lies in the fact that in many problems, mainly for inverse problems for equation (1) with some boundary data under conditions of uncertainty (i.e., insufficiently or inaccurately specified spectral data), such explicit forms are required, at least approximately decision that makes it possible to evaluate the result.

The meaning of the proposed method and the results of the solution are described in the following theorem.

Let

$$\psi(x,\lambda)=\int_0^x (p(t)sin2\lambda t-q(t)cos2\lambda t)dt,$$

$$\varphi(x,\lambda)=\int_0^x p(t)cos2\lambda t+q(t)sin2\lambda tdt,\quad H=H(\lambda x)=\begin{pmatrix} cos\lambda x & -sin\lambda x\\ sin\lambda x & cos\lambda x\end{pmatrix}$$

$$\theta(x)=\int_0^x \varphi' sh\varphi dt,\qquad \vartheta(x)=-\int_0^x \psi' e^{-\psi}dt$$

**Theorem.** The solution to the Dirac matrix equation (1) with initial conditions (2) is described by the following formula

$$Y(x,\lambda,\alpha)=\begin{pmatrix} cos\lambda x & -sin\lambda x\\ sin\lambda x & cos\lambda x\end{pmatrix}\begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix} cos\alpha\\ -sin\alpha\end{pmatrix} \qquad (3)$$

where

$$G_{11}=1+\theta(x)+\vartheta(x)-\int_0^x \psi'\theta dt+\int_0^x \psi'\int_0^{t_1}\theta\psi' d\tau_1 dt+$$

$$+\int_0^x \varphi'\int_0^{t_1}[\varphi'\vartheta(x)+\psi' sh\varphi]d\tau_1 dt-\cdots-\sum_{k=3}^{\infty}\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}\theta\psi' d\tau_k\cdots d\tau_1 dt+\cdots$$

$$+\sum_{k=3}^{\infty}\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}[\varphi'\vartheta(x)+\psi' sh\varphi]d\tau_k\cdots d\tau_2 d\tau_1 dt$$

$$G_{12}=-sh\varphi-\int_0^x[\varphi'\vartheta(x)-\psi' sh\varphi]\,dt-\int_0^x \varphi'\int_0^{t_1}\theta\psi' d\tau_1 dt+$$

$$+\int_0^x \psi'\int_0^{t_1}[\varphi'\vartheta(x)-\psi' sh\varphi]\,d\tau_1 dt-\cdots-\sum_{k=3}^{\infty}(-1)^k(\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}\theta\psi' d\tau_k\cdots d\tau_1 dt$$

$$+\cdots$$

$$+\sum_{k=3}^{\infty}(-1)^k\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}[\varphi'\vartheta(x)+\psi' sh\varphi]d\tau_k\cdots d\tau_2 d\tau_1 dt$$

$$G_{21}=-sh\varphi-\int_0^x[\varphi'\vartheta(x)+\psi' sh\,\varphi]dt+\int_0^x \varphi'\int_0^{t_1}\theta\psi' d\tau_1 dt-$$

$$-\int_0^x \psi'\int_0^{t_1}[\varphi'\vartheta(x)+\psi' sh\,\varphi]d\tau dt-\cdots+$$

$$-\sum_{k=3}^{\infty}(-1)^k(\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}\theta\psi' d\tau_k\cdots d\tau_1 dt+\cdots+$$

$$+\sum_{k=3}^{\infty}(-1)^k\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}[\varphi'(e^{-\psi}-1)+\psi' sh\varphi]d\tau_k\cdots d\tau_2 d\tau_1 dt$$

$$G_{22}=e^{\psi}+\theta(x)+\int_0^x\theta\psi' dt+\int_0^x\psi'\int_0^{t_1}\theta\psi' d\tau_1 dt+$$

$$+\int_0^x\varphi'\int_0^{t_1}[\varphi'\vartheta(x)-\psi' sh\varphi]d\tau_1 dt+\cdots+$$

$$+\sum_{k=3}^{\infty}\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}\theta\psi' d\tau_k\cdots d\tau_1 dt_1+\cdots+$$

$$-\sum_{k=3}^{\infty}\int_0^x f_0\int_0^{t_1} f_1\int_0^{t_2} f_2\cdots\int_0^{t_k}[\varphi'\vartheta(x)+\psi' sh\varphi]d\tau_k\cdots d\tau_2 d\tau_1 dt,$$

here $f_k\in\{\psi,\varphi\}$.. The infinite sums involved in в $F_{ij}$, are absolutely convergent and are equivalent in modulus to $O\left(\frac{M^k}{k!}\right)$, where $M=\max_{0<x<\infty}(e^{\psi},e^{-\psi},e^{\varphi})$. Formula (3) is valid for both small and large values of λ≠0.

**Proof.** Let us introduce the following notation

$$Y(x,\lambda,\alpha)=\begin{pmatrix}cos\lambda x & -sin\lambda x\\ sin\lambda x & cos\lambda x\end{pmatrix}U(x,\lambda,\alpha)$$

Then problem (1)-(2) is reduced to solving the following matrix differential equation

$$U'+\begin{pmatrix}p(t)sin2\lambda t-q(t)cos2\lambda t & p(t)cos2\lambda t+q(t)sin2\lambda t\\ p(t)cos2\lambda t+q(t)sin2\lambda t & p(t)sin2\lambda t-q(t)cos2\lambda t\end{pmatrix}U=0$$

or

$$U'+\begin{pmatrix}\psi' & \varphi'\\ \varphi' & -\psi'\end{pmatrix}U=0 \qquad U=\begin{pmatrix}u_1(x)\\ u_2(x)\end{pmatrix} \tag{3}$$

with initial conditions

$$u_1(0)=cos\alpha, \qquad u_2(0)=-sin\alpha \tag{4}$$

The representation of $\psi$, $\varphi$ is described above.

The integral form of problem (3), (4) is described by the Volterra equation of the 2nd kind

$$U(x,\lambda,\alpha)=\begin{pmatrix}cos\alpha\\ -sin\alpha\end{pmatrix}-\int_0^x\begin{pmatrix}\psi' & \varphi'\\ \varphi' & -\psi'\end{pmatrix}U(t,\lambda,\alpha)dt \tag{5}$$

Problems (3)-(4) or equation (5), provided that the elements of the matrix $\psi',\varphi'$ are uniformly continuous in the finite interval under consideration, have a solution and a unique solution [2]. Let's start solving this integral equation. Equation (5) can be solved using the successive approximation method. As a zero approximation we choose the following

$$\widetilde{U}_0=\begin{pmatrix}cos\alpha\\ -sin\alpha\end{pmatrix}$$

Inserting $\widetilde{U}_0$ together with U in the subintegral (5), we obtain

$$\widetilde{U}_1=\begin{pmatrix}cos\alpha\\ -sin\alpha\end{pmatrix}-\int_0^x\begin{pmatrix}\psi' & \varphi'\\ \varphi' & -\psi'\end{pmatrix}\begin{pmatrix}cos\alpha\\ -sin\alpha\end{pmatrix}dt=\left[I-\begin{pmatrix}\psi & \varphi\\ \varphi & -\psi\end{pmatrix}\right]\begin{pmatrix}cos\alpha\\ -sin\alpha\end{pmatrix}$$

The same thing, for $\widetilde{U}_2$ we get

$$\widetilde{U}_2 = \begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix} - \int_0^x \begin{pmatrix} \psi' & \varphi' \\ \varphi' & -\psi' \end{pmatrix} \widetilde{U}_1 dt =$$

$$= \left[ \begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix} - \int_0^x \begin{pmatrix} \psi' & \varphi' \\ \varphi' & -\psi' \end{pmatrix} \left[ I - \begin{pmatrix} \psi & \varphi \\ \varphi & -\psi \end{pmatrix} \right] \begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix} dt \right] =$$

$$= \left[ I - \begin{pmatrix} \psi & \varphi \\ \varphi & -\psi \end{pmatrix} + \int_0^x \begin{pmatrix} \psi' & \varphi' \\ \varphi' & -\psi' \end{pmatrix} \begin{pmatrix} \psi & \varphi \\ \varphi & -\psi \end{pmatrix} dt \right] \begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix} =$$

$$= \left[ I - \begin{pmatrix} \psi & \varphi \\ \varphi & -\psi \end{pmatrix} + \frac{\psi^2 + \varphi^2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_0^x (\psi'\varphi - \varphi'\psi) dt \right] \begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix}$$

Continuing in a similar way, we get

$$\widetilde{U}_n = [I - A_1 + A_2 - A_3 + \cdots + (-1)^n A_n] \begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix} \qquad (6)$$

where $A_n$ is determined by the following recurrence relations

$$A_{n+1} = \begin{pmatrix} a_{n+1} & (-1)^n b_{n+1} \\ b_{n+1} & -(-1)^n a_{n+1} \end{pmatrix} = \int_0^x \begin{pmatrix} \psi' & \varphi' \\ \varphi' & -\psi' \end{pmatrix} A_n(t) dt =$$

$$= \int_0^x \begin{pmatrix} \psi' a_n + \varphi' b_n & (-1)^n (\varphi' a_n - \psi' b_n) \\ \varphi' a_n - \psi' b_n & -(-1)^n (\psi' a_n + \varphi' b_n) \end{pmatrix} dt,$$

$$a_{n+1} = \int_0^x [\psi' a_n + \varphi' b_n] dt, \qquad b_{n+1} = \int_0^x [\varphi' b_n - \psi' a_n] dt$$

$$a_0 = 1, \qquad b_0 = 0, \qquad n = 0, 1, 2, \cdots$$

For convenience, we group the sequence of the matrix $A_n$ with even and odd indices. Then for n>1 we have

$$A_{2(n+1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[ \frac{\psi^{2(n+1)} + \varphi^{2(n+1)}}{2(n+1)!} + \right.$$

$$+ \int_0^x \psi' \int_0^{t_1} \left( \psi' \frac{\varphi^{2n}}{2n!} + \varphi' \int_0^{t_2} \left( \varphi' \frac{\psi^{2n-1}}{(2n-1)!} - \psi' \frac{\varphi^{2n-1}}{(2n-1)!} \right) d\tau_2 \right) d\tau_1 dt + \cdots$$

$$\left. + \int_0^x \cdots \psi' \int_0^{t_{2n-1}} \left( \psi' \frac{\varphi^2}{2} + \varphi' W(\varphi, \psi) \right) d\tau_{2n-1} \cdots dt \right]$$

$$- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left[ \int_0^x \left( \varphi' \frac{\psi^{2n+1}}{(2n+1)!} - \psi' \frac{\varphi^{2n+1}}{(2n+1)!} \right) dt - \right.$$

$$\left. - \int_0^x \cdots \psi' \int_0^{t_{2n-1}} \left( \varphi' \frac{\psi^2}{2} - \psi' W(\varphi, \psi) \right) d\tau_{2n-1} \cdots d\tau_1 dt \right]$$





Web of Discoveries: Journal of Analysis and Inventions
webofjournals.com/index.php/3

$$A_{2n+1}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\left[\frac{\psi^{2n+1}}{(2n+1)!}+\int_0^x\psi'\frac{\varphi^{2n}}{2n!}d\tau_1+\right.$$
$$+\int_0^x\psi'\int_0^{t_1}\psi'\int_0^{t_2}\left(\psi'\frac{\varphi^{2n}}{2n!}+\varphi'W(\varphi,\psi)\right)d\tau_2\,d\tau_1dt+$$
$$+\int_0^x\psi'\int_0^{t_1}\varphi'\int_0^{t_2}\left(\varphi'\frac{\psi^{2n}}{2n!}-\psi'W(\varphi,\psi)\right)d\tau_2\,d\tau_1dt+$$
$$\left.-\int_0^x\varphi'\int_0^{t_1}\psi'\int_0^{t_2}\left(\varphi'\frac{\psi^{2n}}{2n!}-\psi'W(\varphi,\psi)\right)d\tau_2\,d\tau_1dt\right]+$$
$$+\begin{pmatrix}0&1\\1&0\end{pmatrix}\left[\frac{\psi^{2n+1}}{(2n+1)!}+\int_0^x\varphi'\frac{\psi^{2n}}{2n!}dt-\right.$$
$$\left.+\int_0^x\varphi'\int_0^{t_1}\varphi'\int_0^{t_2}\left(\varphi'\frac{\psi^{2n}}{2n!}-\psi'W(\varphi,\psi)\right)d\tau_2\,d\tau_1dt\right]+\cdots$$
$$\left.-\int_0^x\psi'\int_0^{t_1}\left(\varphi'\int_0^{t_2}\left(\varphi'\frac{\psi^{2n}}{2n!}-\psi'W(\varphi,\psi)\right)d\tau_2\right)d\tau_1dt\right]$$

All elements of the matrix $A_n$ are convergent sequences, which, as n increases, tend to zero in order $O\left(\frac{M^n}{n!}\right)$. We insert all found matrices $A_n$ into (6). For convenience, we introduce the following notation

$$I-A_1+A_2-\cdots+(-1)^nA_n+\cdots=\begin{pmatrix}G_{11}&G_{12}\\G_{21}&G_{22}\end{pmatrix}\qquad(6')$$

Since the infinite series involved in the right side of equality (6') is convergent [9], as n→∞, we have

$$G_{11}=\sum_{i=0}^{\infty}a_{2i}-\sum_{i=0}^{\infty}a_{2i+1}=1-\ \psi++\sum_{i=1}^{\infty}\frac{\psi^{2i}+\varphi^{2i}}{(2i)!}-$$
$$-\sum_{i=2}^{\infty}\left\{\frac{\psi^{2i-1}}{(2i-1)!}+\int_0^x\frac{\psi'\varphi^{2(i-1)}}{(2i-2)!}dt+\cdots\right\}=e^{-\psi}+\theta(x)+\int_0^x\psi'\theta(x)\,dt+\cdots$$
$$G_{12}=\sum_{i=0}^{\infty}b_{2i}-\sum_{i=0}^{\infty}b_{2i+1}=\sum_{i=1}^{\infty}\left\{\frac{1}{(2i-1)!}\int_0^x\{\psi'\varphi^{2i-1}-\varphi'\psi^{2i-1}\}dt+\cdots\right\}-$$
$$-\sum_{i=1}^{\infty}\frac{\varphi^{2i-1}}{(2i-1)!}-\sum_{i=2}^{\infty}\int_0^x\frac{\varphi'\left(\psi^{2(i-1)}\right)dt}{(2i-2)!}+\cdots==-sh\varphi+\int_0^x[\psi'sh(\varphi)-\varphi'\vartheta(x)]\,dt+\cdots$$
$$G_{21}=\sum_{i=0}^{\infty}b_{2i}-\sum_{i=0}^{\infty}b_{2i+1}=\sum_{i=1}^{\infty}\left\{\frac{1}{(2i-1)!}\int_0^x(\varphi'\psi^{2i-1}-\psi'\varphi^{2i-1})dt+\cdots\right\}-$$





Web of Discoveries: Journal of Analysis and Inventions
webofjournals.com/index.php/3

$$-\sum_{i=1}^{\infty}\frac{\varphi^{2i-1}}{(2i-1)!}-\sum_{n=2}^{\infty}\int_{0}^{x}\frac{[\varphi'(\psi^{2(n-1)})]dt}{(2n-2)!}+\cdots==-sh\varphi-\int_{0}^{x}[\varphi'\vartheta(x)+\psi' sh\varphi]dt+\cdots,$$

$$G_{22}=\sum_{i=0}^{\infty}a_{2i}-\sum_{i=0}^{\infty}a_{2i+1}=1+\psi+\sum_{i=1}^{\infty}\left\{\frac{\psi^{2i}+\varphi^{2i}}{(2i)!}+\cdots\right\}+$$

$$+\sum_{i=2}^{\infty}\left\{\frac{\psi^{2i-1}}{(2i-1)!}+\int_{0}^{x}\frac{[\psi'\varphi^{2(i-1)}]}{(2n-2)!}dt+\cdots\right\}=e^{\psi}+\theta(x)+\int_{0}^{x}\psi'\theta(x)\,dt\cdots.$$

The theorem is proved.

By simplifying (3) for sufficiently large λ, one can obtain all the classical formulas for problem (1)-(2), such as asymptotic formulas for eigenvalues λ, asymptotic formulas for eigenfunctions, asymptotic formulas for normalizing numbers, etc. [1].

**References**


1. Левитан Б.М., Саргсян И.С. Операторы Штурма-Лиувилля и Дирака. 1988. 432 с. ISBN 5-02-013751-0.
2. Абдуназаров Р. A.Mamatov Численное решение обратной задачи для системы Дирака. International scientific journal *Interpretation and Researches*, *1*(5). Page 130-135. ISSN:2181-4163
3. Рабимкул, А., Иброҳимов , Ж. Б. ў., Пўлатов, Б. С., & Нориева, А. Ж. қ. (2023). Аргументларни гуруҳларга ажратиб баҳолаш усулида кўп параметрли ночизиқли регрессия тенгламаларини қуриш масалалари. *Educational Research in Universal Sciences*, *2*(2), 174–178. Retrieved from https://erus.uz/index.php/er/article/view/1704
4. R.Abdunazarov. Problems of constructing an efficient computational algorithm for restoring the parameters of the Sturm-Louisville operator. international journal of contemporary scientific and technical research.2181-3884. 2023 in jbnuu. page 80-86/
5. Р.Абдуназаров. Bir oʻlchovli Dirak tenglamasi uchun teskari masalani yechishda tezkor hisoblash algoritmini qurish masalalari. “Komputer ilmlari va dasturlash texnologiyalari” mavzusidagi xalqaro ilmiy texnik konferensiya materiallari. Jizzax 2023 yil 13 oktabr. T.1.№2 186-191 бет
6. Абуназаров Р. Po’latov Baxtiyor, Noriyeva Aziza, Xolmanova Klara. Восстановления ядро спектральной меры оператора Дирака по неточным спектральным данным. Zamonaviy innovatsion tadqiqotlarning dolzarb muammolari va rivojlanish tendensiyalari: yechimlar va istiqbollar mavzusidagi Respublika ilmiy-texnik anjumani. 27.05.2023. 258-262 betlar.
7. А.М.Тер-Крикоров. Матричные функции и линейные дифференциальные уравнения. Уч-метод. пособие/ сост.: А.М.Крикоров. М.: МФТИ, 2014 -42 с.
8. 1. Шарипова С. МАТЕМАТИКА В БИОЛОГИИ //Conferencea. – 2022. – С. 196-199.
9. 2. Sharipova S., INTEGRALLAR N. M. G. E. K. YORDAMIDA YECHISHGA DOIR MASALALAR //Журнал математики и информатики. – 2022. – Т. 2. – №. 1.
10. 3. Шарипова С. Ф., Олтмишев А. СОВРЕМЕННЫЕ ПЕДАГОГИЧЕСКИЕ ТЕХНОЛОГИИ В ОБУЧЕНИИ МАТЕМАТИКЕ. – 2022.

11. 4. Sharipov X., Sharipova S. ORBITS ARE A FAMILY OF VECTOR FIELDS AND A HYPERBOLIC PARABOLOID //International Journal of Contemporary Scientific and Technical Research. – 2023. – Т. 1. – №. 2. – С. 50-57.
12. Шарипова С. Ф., Бахриддинова А. ПРИМЕНЕНИЕ СКАЛЯРНОГО ПРОИЗВЕДЕНИЯ ВЕКТОРОВ ДЛЯ РЕШЕНИЯ УРАВНЕНИЙ И СИСТЕМ //Ta'limning zamonaviy transformatsiyasi. – 2024. – Т. 6. – №. 2. – С. 175-185.