Web of Teachers: Inderscience Research
webofjournals.com/index.php/1

HIGH-ORDER MOMENTS OF THE SUM OF RANDOM VARIABLES USING THE METHOD OF CHARACTERISTIC FUNCTIONS

Dushatov N. T.
Almalyk Branch of Tashkent State Technical University Named After Islam Karimov
n_dushatov@rambler.ru

Abstract

The problem of determining the order moment of the sum of random variables (r.v.) using the method of characteristic functions (ch.f.) is studied in the article. order moment of the sum of independent and dependent r.v.-es subject to the exponential law with the parameter is found.

Keywords: Random variable, density function, exponential distribution, characteristic function, order moment, correlation coefficient, numerical covariance coefficient, conjugate density function.

Introduction

1. Introduction. Definition 1. [1] Let ξ be a random variable. Let $i=\sqrt{-1}$ be a imaginary unit. The function $f:R\to C$ defined by

$$f_\xi(t)=Me^{itx}=\int_\Omega e^{itx}dP=\int_{-\infty}^{\infty}e^{itx}dF_\xi(x) \qquad (1)$$

is called the characteristic function of ξ.

If ξ discrete r.v., then ch.f.

$$f_\xi(t)=\sum_k e^{itx_k}P(\xi=x_k)=\sum_k e^{itx_k}p(x_k). \qquad (2)$$

If the random variable ξ has an absolutely continuous distribution, and $p(x)$ is its density function, then

$$f_\xi(t)=\int_{-\infty}^{\infty}e^{itx}p(x)dx.$$

Theorem 1. [1] Let $f_\xi(t)$ is characteristic function of r.v. ξ. Then

1. $f_\xi(0)=1$, $|f_\xi(t)|\le 1$;
2. For any a and b constants we have $f_{a+b\xi}(t)=e^{iat}f_\xi(bt)$;
3. $\bar{f}_\xi(t)=f_\xi(-t)=f_{-\xi}(t)$;

4. If $\xi_1, \xi_2, ..., \xi_n$ are independent r.v.-es, then

$$f_{\xi_1+...+\xi_n}(t) = f_{\xi_1}(t) \cdot f_{\xi_2}(t) \cdots f_{\xi_n}(t).$$

5. Ch.f. $f_\xi(t)$ is uniformly continuous in $R_1 = (-\infty, \infty)$.

6. For the characteristic function $f_\xi(t)$ to be valid, it is necessary and sufficient for the distribution function of the random variable ξ to be symmetric.

Theorem 2. [1] If $E|\xi|^k < \infty$ and the characteristic function $f_\xi(t)$ has continuous derivatives up to order k, and the following relations hold:

$$f_\xi^{(\nu)}(t) = i^\nu \int_{-\infty}^{\infty} x^\nu e^{itx} dF_\xi(x), \quad \nu = 1,2,...,k, \tag{3}$$

$$f_\xi^{(\nu)}(0) = i^\nu E\xi^\nu, \tag{4}$$

$$f_\xi(t) = \sum_{\nu=0}^{k} \frac{(it)^\nu}{\nu!} E\xi^\nu + o(t^k), \quad t \to 0. \tag{5}$$

Definition 2. [1] Let (Ω, A, P) be a probability space, and let ξ be a random variable defined on this space, with $k > 0$ being a given number. If the mathematical expectation $M|\xi|^k$ exists, then the value $a_k = M\xi^k$ is called the $k-$th order raw moment of the random variable ξ, and the value $m_k = M|\xi|^k$ is called the $k-$th order absolute moment.

2. Independent case.

If $\xi_1, \xi_2, ..., \xi_n$ are independent and identically distributed random variables following an exponential distribution with parameter α, then their distribution is given by $P(\xi_i \le x) = 1 - e^{-\alpha x}, \ x \ge 0, \ i = 1, 2, ...$ [2].

Theorem 3. If $\xi_1, \xi_2, ..., \xi_n$ are independent random variables following the exponential distribution with parameter α, then for any $k, n = 1, 2, ...$ we have

$$M\left(\xi_1 + \xi_2 + ... + \xi_n\right)^k = \frac{n(n+1)(n+2)...(n+k-1)}{\alpha^k}.$$

Proof. The characteristic function of the exponential distribution is $f_{\xi_i}(t) = \frac{\alpha}{\alpha - it}$. Let us denote $\eta_n = \xi_1 + \xi_2 + ... + \xi_n$. The main problem is to find $M\eta_n{}^k$. According to relation (4), we have $M\eta_n{}^k = \frac{1}{i^k} f_{\eta_n}^{(k)}(0)$.

Using the property of characteristic functions from Theorem 1, we get:

$$f_{\eta_n}(t)=\prod_{j=1}^{n} f_{\xi_j}(t)=\left(\frac{\alpha}{\alpha-it}\right)^n .$$

Now we compute its $k-$ th derivative with respect to t:

$$f_{\eta_n}^{'}(t)=\alpha^n(-n)(\alpha-it)^{-n-1}(-i)=\alpha^n ni(\alpha-it)^{-n-1},$$

$$f_{\eta_n}^{''}(t)=\alpha^n n(-n-1)i(\alpha-it)^{-n-2}(-i)=\alpha^n n(n+1)i^2(\alpha-it)^{-n-2},$$

....

$$f_{\eta_n}^{(k)}(t)=\alpha^n n(n+1)(n+2)...(n+k-1)i^k(\alpha-it)^{-n-k},$$

$$f_{\eta_n}^{(k)}(t)=\frac{\alpha^n n(n+1)(n+2)...(n+k-1)i^k}{(\alpha-it)^{n+k}}.$$

Now we compute its $k-$ th derivative with respect to $t=0$:

$$f_{\eta_n}^{(k)}(0)=\frac{n(n+1)(n+2)...(n+k-1)i^k}{\alpha^k}$$

Then,

$$M\eta_n{}^k=\frac{1}{i^k}f_{\eta_n}^{(k)}(0)=\frac{1}{i^k}\cdot\frac{n(n+1)(n+2)...(n+k-1)i^k}{\alpha^k}=\frac{n(n+1)(n+2)...(n+k-1)}{\alpha^k}.$$

3. Dependent case.

Characteristic function of the sum of dependent random variables

$$f_{\eta_n}(t)=E\left(e^{it(\xi_1+\xi_2+...+\xi_n)}\right).$$

When correlation exists, this changes to:

$$f_{\eta_n}(t)=\int_{R^n} e^{it(x_1+x_2+...+x_n)} f_{\xi_1,\xi_2,...,\xi_n}(x_1,x_2,...,x_n)dx_1dx_2...dx_n ,$$

where the conjugate density function of random variables $f_{\xi_1,\xi_2,...,\xi_n}(x_1,x_2,...,x_n)$ depends on the correlation.

If even random variables (ξ_1,ξ_2) are correlated, then their conjugate density function

$$f_{\xi_1,\xi_2}(x_1,x_2)=\alpha^2 e^{-\alpha(x_1+x_2)}\cdot g(\rho,x_1,x_2),$$

where $g(\rho,x_1,x_2)-$ a function describing a relationship, e.g.

$$g(\rho,x_1,x_2)=1+\rho\cdot h(x_1,x_2),$$

$h(x_1,x_2)-$ determined by the nature of the correlation coefficient. Depending on the type of correlation, $g(\rho,x_1,x_2)$ has a different shape.

If $\eta_2=\xi_1+\xi_2$ and $\xi_1,\ \xi_2$ are correlated, then

$$E\eta_2^n=\sum_{i=0}^{n} C_n^i E[\xi_1^i\xi_2^{n-i}],$$

here, even moments are written as follows.

$$E[\xi_1^i \xi_2^{n-i}] = \rho \cdot \sqrt{E\xi_1^i E\xi_2^{n-i}}\ .$$

Let $\xi_1, \xi_2, ..., \xi_n$ be dependent random variables with correlation coefficient

$$\rho_{ij} = \frac{Cov(\xi_i, \xi_j)}{\sqrt{Var\xi_i}\sqrt{Var\xi_j}}\ .$$

Then the first and second order moments are:

$$E[\xi_i] = \frac{1}{\alpha},\ Var[\xi_i] = \frac{1}{\alpha^2},\ Cov\left(\xi_i, \xi_j\right) = \rho_{ij} \cdot \frac{1}{\alpha^2},$$

where ρ_{ij} – are the numerical covariance coefficients of r.v.-s ξ_i and ξ_j

Joint mathematical expectation for two variables

$$E(\xi_i \xi_j) = E(\xi_i) \cdot E(\xi_j) + Cov(\xi_i, \xi_j)$$

Now we determine the moments of the sum.

For $k = 1$, it is evident that the result easily follows from the linearity property of expectation, both in the dependent and independent cases.

$$E\left(\xi_1 + \xi_2 + ... + \xi_n\right) = E\left(\xi_1\right) + E\left(\xi_2\right) + ... + E\left(\xi_n\right) = \frac{n}{\alpha}.$$

For $k = 2$,

$$E\left(S_n^2\right) = \sum_{i=1}^{n} E\xi_i^2 + 2\sum_{1 \le i < j \le n} E\left(\xi_i \xi_j\right),$$

where $E\xi_i^2 = Var(\xi_i) + (E\xi_i)^2 = \dfrac{2}{\alpha^2},\ E\left(\xi_i \xi_j\right) = \dfrac{1}{\alpha^2} + \rho_{ij} \cdot \dfrac{1}{\alpha^2}$.

Considering the above

$$E\left(S_n^2\right) = n \cdot \frac{2}{\alpha^2} + 2 \cdot \frac{1}{\alpha^2} \cdot \sum_{1 \le i < j \le n} (1 + \rho_{ij}).$$

References

1. A.A.Borovkov. Mathematical Statistics. Taylor&Francis. 1998.
2. Dushatov N.T., Obloqulov S.Z. Tasodifiy miqdorlar yigindisining yuqori tartibli momentlarini xarakteristik funksiyalar metodi orqali aniqlash. Oriental Renaissance: Innovative, educational, natural and social sciences. *4(6), June, 2024.pp. 128-130.*
3. Б.А.Севостьянов, В.И.Чистяков, А.М.Зубков «Сборник задач по теории вероятностей», Москва, «Наука», 1989 г.