ON THE INTEGRATION OF SOME TRIGONOMETRIC FUNCTIONS

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Abstract
This article systematically covers the theoretical foundations and practical methods of integrating trigonometric functions. The main attention is paid to the integration of sine, cosine, tangent, cotangent and their expressions in power and product forms, the article shows the procedure for effective use of trigonometric functions, the method of changing variables. Each method is reinforced with step-by-step solved examples.

Keywords: Trigonometric functions, integration, indefinite integral, changing variables, trigonometric functions, integral calculus.

Introduction

BA'ZI TRIGONOMETRIK FUNKSIYALARNI INTEGRALLASH HAQIDA.

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Annotatsiya:
Ushbu maqolada trigonometrik funksiyalarni integrallashning nazariy asoslari va amaliy usullari tizimli ravishda yoritilgan. Asosiy e'tibor sinus, kosinus, tangens, kotangens va ularning darajali hamda ko'paytma shakllaridagi ifodalarini integrallashga qaratilgan bo'lib, maqolada trigonometrik ayniyatlar, o'zgaruvchini almashtirish usuli samarali foydalanish tartibi ko'rsatilgan. Har bir usul bosqichma-bosqich yechilgan misollar bilan mustahkamlangan.

Kalit so'zlar: trigonometrik funksiyalar, integrallash, aniqmas integral, o'zgaruvchini almashtirish, trigonometrik ayniyatlar, integral hisob.

1-Ta'rif. Agar [a;b] segmentning hamma nuqtalarida $F'(x)=f(x)$ tenglik bajarilsa, $F(x)$ funksiya shu segmentda $f(x)$ funksiyaga nisbatan boshlang'ich funksiya deb ataladi.

1-Teorema. Agar $F_1(x)$ va $F_2(x)$ funksiyalar f(x) funksiyaning [a; b] kesmada boshlang'ich funksiyalari bo'lsa, ular orasidagi ayirma o'zgarmas songa teng bo'ladi.

2-Ta'rif. Agar F(x) funksiya [a; b] da f(x) funksiya uchun boshlang'ich funksiya bo'lsa, u holda F(x)+C, (C-const) boshlang'ich funksiyalar to'plamiga f(x) funksiyaning [a; b] dagi aniqmas integrali deyiladi va $\int f(x)dx$ kabi belgilanadi. Bunda $\int$ - integral belgisi, f(x) integral ostidagi funksiya, f(x)dx esa integral ostidagi ifoda deyiladi.

$$\int f(x)dx = F(x) + C \qquad (1)$$

I. $\int$R (sinx, cosx)dx **ko'rinishdagi integral**

Bunda R-rasional funksiya. Bu integral $tg\frac{x}{2} = t$ almashtirish bilan rasional funksiyaning integraliga keltirilishi mumkin.

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2tg\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

$$tg\frac{x}{2} = t \Rightarrow \frac{x}{2} = arctgt \Rightarrow x = 2arctgt \Rightarrow dx = \frac{2dt}{1+t^2};$$

U holda berilgan integral ushbu ko'rinishdagi rasional funksiyaning integraliga keladi, ya'ni

$$\int R(\sin x,\ \cos x)dx = \int R\left[\frac{2t}{1+t^2}; \frac{1-t^2}{1+t^2}\right] \cdot \frac{2dt}{1+t^2} = \int R^*(t)dt, \qquad (2)$$

bunda $R^*(t)$ funksiya t ning rasional funksiyasi.

1-Misol. $\int \frac{dx}{\cos x}$ integralni hisoblang.

Yechilishi. Yuqoridagi yozilgan formulalarga asosan:

$$\int \frac{dx}{\cos x} = \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1-t^2} = \ln\left|\frac{t-1}{t+1}\right| + C = \ln\left|\frac{tg\frac{x}{2} - 1}{tg\frac{x}{2} + 1}\right| + C.$$

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$tg\frac{x}{2}=t \quad (-\pi < x < \pi)$ almashtirishga universal almashtirish deyiladi. Bu almashtirish R(sinx,cosx) ko‘rinishdagi har qanday funksiyani integrallash uchun imkon beradi. Lekin praktikada bu almashtirish ko‘pincha ancha murakkab rasional funksiyaga olib keladi. Shuning uchun universal almashtirish bilan bir qatorda ba’zi hollar uchun maqsadga tez olib keladigan boshqa almashtirishlarni ham bilish foydalidir.

1. Agar R(sinx, cosx) funksiya sinx ga nisbatan toq bo‘lsa, ya’ni R(-sinx, cosx) = - R(sinx, cosx) bo‘lsa, u holda cosx=t, - sinxdx=dt almashtirish bilan rasional funksiyani integraliga keladi.

2.Agar R(sinx, cosx) funksiya cos x ga nisbatan toq bo‘lsa, ya’ni R(sinx, -cosx)=-R(sinx, cosx) bo‘lsa, u holda sinx=t almashtirish bilan rasional funksiyani integraliga keltiriladi.

3.Agar R(sinx, cosx) funksiya sinx va cosx ga nisbatan juft bo‘lsa, ya’ni R(-sinx,-cosx)=R(sinx, cosx) bo‘lsa, u holda tgx=t almashtirish bilan rasionallashtiriladi. Bu holda

$$\sin^2 x=\frac{tg^2x}{1+tg^2x}=\frac{t^2}{1+t^2},\quad \cos^2 x=\frac{1}{1+tg^2x}=\frac{1}{1+t^2},\qquad dx=\frac{dt}{1+t^2}.$$

2-Misol. $\int\frac{\cos^3 x}{1+\sin x}dx$ integral hisoblansin.

Yechilishi: $R(\sin x,\ \cos x)=\frac{\cos^3 x}{1+\sin x}$ funksiya sinx ga nisbatan toq funksiya. Bu integralni $\int R(\sin x)\cos x dx$ ko‘rinishga keltirish oson. Haqiqatdan,

$$\int\frac{\cos^3 x}{1+\sin x}dx=\int\frac{\cos^2 x\cos x}{1+\sin x}dx=\int\frac{1-\sin^2 x}{1+\sin x}\cdot\cos x dx.$$

sinx=t almashtirishni bajaramiz. Bu holda cosxdx=dt. Demak,

$$\int\frac{\cos^3 x}{1+\sin x}dx=\int\frac{1-t^2}{1+t}dt=-\int\frac{t^2-1}{1+t}dt=-\int(t-1)dt=-\frac{t^2}{2}+t+C=$$

$$=-\frac{\sin^2 x}{2}+\sin x+C.$$

3-Misol. $\int\frac{dx}{9-\sin^2 x}dx$ integral hisoblansin.

Yechilishi: tgx=t almashtirishni bajaramiz. Bu holda:

$$\int \frac{dx}{9-\sin^2 x} = \int \frac{dt}{(9-\frac{t^2}{1+t^2})(1+t^2)} = \int \frac{dt}{9+t^2} = \frac{1}{3} arctg\frac{t}{3} + C = \frac{1}{3} arctg\frac{tgx}{3} + C.$$

II. $\int \sin^m x \cdot \cos^n xdx$ **ko'rinishdagi integral.**

Bunda uchta holni qarashga to'g'ri keladi:

a)Berilgan integralda m va n larning kamida bittasi toq son. Aniqlik uchun n-toq son bo'lsin. n=2p+1 deb olib, integralni o'zgartiramiz:

$$\int \sin^m x \cos^{2p+1} xdx = \int \sin^m x \cos^{2p} x \cos xdx =$$

$$\int \sin^m x(1-\sin^2 x)^p \cos xdx.$$

O'zgaruvchini almashtiramiz: $\sin x = t \Rightarrow \cos xdx = dt.$ U holda $\int \sin^m x \cos^n xdx = \int t^m (1-t^2)^p dt,$ bu esa t ning rasional funksiyasining integralidir.

4-Misol. $\int \sin^3 x \cdot \cos^{2026} xdx$ integral hisoblansin.

Yechilishi:

$$\int \sin^3 x \cdot \cos^{2026} xdx = \int \sin^2 x \cdot \sin x \cdot \cos^{2026} xdx = \int (1-\cos^2 x) \cdot \cos^{2026} x \cdot \sin xdx.$$

$\cos x = t$ deymiz, $\sin xdx = -dt,$ u holda

$$\int \sin^3 x \cdot \cos^{2026} xdx = -\int (1-t^2)t^{2026} dt = \int (t^{2028} - t^{2026})dt = \frac{t^{2029}}{2029} - \frac{t^{2027}}{2027} + C$$

$$= \frac{1}{2029}\cos^{2029} x - \frac{1}{2027}\cos^{2027} x + C.$$

b) Berilgan integralda m va n-manfiy bo'lmagan juft son. m=2p, n=2q deymiz. Trigonometriyadan ma'lum bo'lgan formulalarni yozamiz:

$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x).$ Bularni berilgan integralga qo'yamiz:

$$\int \sin^{2p} x \cdot \cos^{2q} xdx = \int (\frac{1}{2} - \frac{1}{2}\cos 2x)^p \cdot (\frac{1}{2} + \frac{1}{2}\cos 2x)^q dx. \quad (3)$$

Darajaga ko'tarib hamda qavslarni ochib, cos2x ning juft va toq darajalarini o'z ichiga olgan hadlarni hosil qilamiz.

Toq darajali hadlar a) holda ko'rsatilgandek integrallanadi. Darajaning juft ko'rsatkichlarini (3) formulalarga ko'ra yana pasaytiramiz. Daraja ko'rsatkichlarni pasaytirishni oson integrallanadigan $\int \cos kxdx$ ko'rinishdagi hadlar hosil bo'lguncha shunday davom ettiramiz.

5-misol. $\int \sin^4 xdx$ integral hisoblansin.

Yechilishi: $\int \sin^4 x dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)dx =$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x dx + \frac{1}{4}\int \frac{1+\cos 4x}{2} dx = \frac{1}{4}x - \frac{1}{4}\sin 2x +$$

$$+ \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

c) Agar ikkala daraja ko‘rsatkich ham juft bo‘lib, ulardan kamida bittasi manfiy bo‘lsa, yuqorida bayon qilingan usul maqsadga olib kelmaydi. Bunda tgx=t (yoki ctgx=t) almashtirishni bajarishga to‘g‘ri keladi.

III. $\int \sin mx \cos nx dx, \int \cos mx \cos nx dx, \int \sin mx \sin nx dx$

ko‘rinishdagi integrallar.

Bular quyidagi formulalar yordamida hisoblanadi:

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x], \qquad (4)$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x], \qquad (5)$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]. \qquad (6)$$

6-Misol. $\int \cos 2025x \cos 2026x dx$ integral hisoblansin.

Yechilishi: (5) formulaga ko‘ra:

$$\cos 2025x \cdot \cos 2026x = \frac{1}{2}[\cos(2025x + 2026x) + \cos(2026x - 2025x)] =$$

$$\frac{1}{2}(\cos 4051x + \cos x).$$

$$\int \cos 2025x \cos 2026x dx = \frac{1}{2}\int (\cos 4051x + \cos x)dx = \frac{1}{2}\cdot\frac{1}{4051}\sin 4051x +$$

$$\frac{1}{2}\cdot \sin x + C = \frac{1}{8102}\sin 4051x + \frac{1}{2}\sin x + C.$$

Xulosa

Xulosa qilib aytganda, murakkab trigonometrik ifodalarni integrallashda fundamental trigonometrik ayniyatlardan to‘g‘ri foydalanish masalani sodda algebraik ko‘rinishga keltirishning eng qisqa yo‘lidir. Trigonometrik funksiyalarni integrallash jarayoni nafaqat matematik hisob-kitoblarni, balki funksiyaning juft-toqligi va darajasiga qarab to‘g‘ri usulni tanlashni talab etadi. Maqolada keltirilgan misollar shuni ko‘rsatadiki: Ratsional trigonometrik funksiyalar uchun esa universal o‘rinlashtirishlar eng universal yechim hisoblanadi. Bu usullarni o‘zlashtirish fizika va muhandislikning tebranishlar va to‘lqinlar jarayonlarini modellashtirish bilan bog‘liq bo‘limlarida muhim ahamiyatga ega.

References

1. B.B.Qarshiyev., Z.Absamatov, Педагогическая деятельность как явление социалъно – педагогической. Научно-методический журнал Проблемы педагогики № 2 (47),2020 С. 100-103.
2. B.B.Qarshiyev., Z.Absamatov, Формирование и совершенствование педагогических навыков в процессе обучения. Научно-методический журнал Вестник науки и образавания 2020. № 6. (84) Часть 1 С 79-84
3. Farhod Halimjonovich Haydarov, Shamshod Akhtamaliyev, Madalixon Nazirov, Behzod Boyxonovich Qarshiyev, Uniqueness of Gibbs measures for an Ising model with continuous spin values on a Cayley tree. Reports on Mathematical Physics.Volume 86, Issue 3, December 2020, Pages 293-302
4. Х. Абдурасулов, И.А.Ачилов, Б.Б.Каршиев. “Применение Бета и Гамма функцией к вычислению некото-рых важных в прикладных задачах” CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND CO MPUTER SCIENCES 49-56 bet.05 | May 2022.
5. X.Abdurasulov,K.N.Xolov,B.B.Qarshiyev. “BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMA YORDAMIDA ELEKTRO-TEXNIKAGA OID MASALALARNI YECHISH”. Oriental Renaissance: Innovative, educational, natural and social ciences. May 2022: 390-398.pdf
6. Б.Э .Эшматов, З.А. Абсаматов, Б.Б. Каршиев, О НЕКОТОРЫХ ЗАДАЧАХ ДЛЯ УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА - SCIENCE AND WORLD, 2013
7. Karshiyev Begzod Boykhanovich. (2023). DYNAMIC SYSTEMS AND SOME OF THEIR APPLICATIONS. INTERNATIONAL BULLETIN OF APPLIED SCIENCE AND TECHNOLOGY, 3(5), 495–497. https://doi.org/10.5281/zenodo.7939563
8. B.B. Qarshiyev. MAIN DIRECTIONS IN THE DEVELOPMENT OF STEAM EDUCATION ELEMENTS. INTERNATIONAL BULLETIN OF APPLIED SCIENCE AND TECHNOLOGY, 3(5), 603–609. https://doi.org/10.5281/zenodo.7949798 (2023).
9. BB Qarshiyev . APPLICATION OF STEAM IN INSTITUTIONS OF EDUCATION was published by in the International Journal of Education, Social Science & Humanities (Vol. 12, Issue 5, pp. 870-874) in 2024.