

MATHEMATICAL MODELING OF STABILITY AND BIFURCATION IN LOTKA–VOLTERRA SYSTEMS UNDER PARAMETRIC UNCERTAINTY

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Abstract

This paper develops a mathematical framework for studying the stability and bifurcation behavior of Lotka–Volterra dynamical systems in the presence of parametric uncertainty. The growth, mortality, self-limitation, and interaction coefficients are represented by nominal values with bounded perturbations. For a modified predator–prey Lotka–Volterra system, the positive coexistence equilibrium, feasibility condition, Jacobian matrix, trace–determinant stability criteria, sensitivity coefficients, and a robust stability margin are derived. The critical surface is shown to separate the coexistence regime from the loss of a biologically feasible positive equilibrium. A numerical experiment demonstrates how admissible parameter variation shifts the equilibrium and reduces the stability margin. The analysis also clarifies that, for the selected continuous self-limited model with positive parameters, an interior Hopf bifurcation cannot occur without an additional feedback mechanism; therefore, richer bifurcation scenarios require extensions such as delay, saturation, harvesting, or discrete-time updating. The proposed approach provides a reproducible basis for uncertainty-aware analysis of ecological and related Lotka–Volterra-type models.

Keywords: Lotka–Volterra system, parametric uncertainty, mathematical modeling, stability, bifurcation, Jacobian matrix, sensitivity analysis, robust stability.

Introduction

Lotka–Volterra models are classical nonlinear models for interacting populations and remain an important basis for mathematical ecology and dynamical-systems theory [1–4]. In practical applications, however, the parameters of such models are not known exactly. Growth and mortality rates vary with environmental conditions, while interaction coefficients are estimated from finite observations. Therefore, conclusions obtained at one nominal parameter vector may fail when admissible uncertainty is taken into account.

The purpose of this work is to construct an uncertainty-aware modeling procedure connecting parameter variation with equilibrium displacement, local stability, sensitivity, and critical transitions. The scientific contribution is the explicit combination of bounded parametric uncertainty with a stability-margin formulation for a self-limited Lotka–Volterra model, together with an analytical separation of feasibility loss from genuine Hopf-type instability.

2. Mathematical model and uncertainty set

We consider the modified two-species Lotka–Volterra predator–prey model

$$\frac{dx}{dt} = x(r - kx - ay), \quad \frac{dy}{dt} = y(-m + bx). \quad (1)$$

Here $x(t)$ and $y(t)$ denote prey and predator densities, respectively; $r > 0$ is the intrinsic prey growth rate, $k > 0$ is the prey self-limitation coefficient, $a > 0$ is the predation coefficient, $m > 0$ is the predator mortality rate, and $b > 0$ is the conversion coefficient.

The uncertain parameters are represented as

$$p_i = p_i^0 \cdot (1 + \delta_i), \quad |\delta_i| \leq \varepsilon_i, \quad p = (r, k, a, m, b).$$

Thus each parameter belongs to an interval $p_i \in [p_i^-, p_i^+]$ and the admissible uncertainty set U is a hyper-rectangle in parameter space. This representation is suitable for interval evaluation, parameter sweeps, and Monte Carlo sampling.

3. Equilibrium and feasibility conditions

Setting the right-hand sides of (1) equal to zero gives the positive coexistence equilibrium

$$x^* = \frac{m}{b}, \quad y^* = \frac{r - \frac{\kappa m}{b}}{a}. \quad (2)$$

The condition $y^* > 0$ is equivalent to

$$F(p) = rb - \kappa m > 0. \quad (3)$$

The scalar $F(p)$ is used as a feasibility margin. Under bounded uncertainty, a sufficient condition for robust coexistence is

$$r^- b^- - \kappa^+ m^+ > 0. \quad (4)$$

Condition (4) guarantees that every admissible parameter realization retains a biologically meaningful positive equilibrium.

4. Local stability analysis

The Jacobian matrix of system (1) is

$$J(x, y) = \begin{pmatrix} r - 2\kappa x - ay & -ax \\ by & -m + bx \end{pmatrix}.$$

Using the equilibrium relations, at the positive equilibrium we obtain

$$J^* = \begin{pmatrix} -\kappa x^* & -ax^* \\ by^* & 0 \end{pmatrix}.$$

Therefore

$$\text{tr}(J^*) = -\frac{\kappa m}{b} < 0, \quad \det(J^*) = \frac{m(rb - \kappa m)}{b}.$$

For a two-dimensional continuous system, local asymptotic stability follows from $\text{tr}(J^*) < 0$ and $\det(J^*) > 0$. Hence, in the present model, every feasible positive equilibrium satisfying (3) is locally asymptotically stable. The discriminant

$$D = [\text{tr}(J^*)]^2 - 4\det(J^*).$$

distinguishes monotone and oscillatory convergence. If $D < 0$, the equilibrium is a stable focus; if $D > 0$, it is a stable node.

5. Robust stability and bifurcation criterion

To quantify the distance from a critical state, define the local stability margin

$$M(p) = \min \left\{ -\text{tr}(J^*(p)), \det(J^*(p)) \right\}.$$

The robust stability margin over the uncertainty set U is

$$M_{\text{rob}} = \min_{p \in U} M(p).$$

If $M_{\text{rob}} > 0$, all admissible realizations preserve local asymptotic stability. If M_{rob} approaches zero, the uncertainty set reaches a critical boundary. For the present model, the principal critical surface is

$$rb - \kappa m = 0. \quad (5)$$

At (5), the positive coexistence equilibrium loses biological feasibility. This is different from a Hopf bifurcation. A Hopf bifurcation in a general planar continuous system requires

$\text{tr}(J^*) = 0$, $\det(J^*) > 0$ and a transversality condition [5]. Since $\text{tr}(J^*) = -\frac{\kappa m}{b} < 0$ for

positive k, m, b , the selected model cannot undergo an interior Hopf bifurcation solely by varying these positive parameters. This distinction prevents an incorrect interpretation of feasibility loss as oscillatory instability.

6. Sensitivity analysis

For an equilibrium quantity $z^* = z^*(p)$, the normalized sensitivity coefficient is defined by

$$S_p^z = \frac{p}{z^*} \frac{\partial z^*}{\partial p}.$$

For $x^* = \frac{m}{p}$, one obtains $S_m^x = 1$ and $S_b^x = -1$. Therefore, small relative changes in m and b

produce equal-magnitude, opposite-direction relative changes in the equilibrium prey density. For y^* , the sensitivity increases when $F(p) = rb - \kappa m$ approaches zero, indicating that parameter estimation becomes especially important near the critical surface.

7. Numerical experiment

Consider the nominal parameter vector

$$r = 1.20, \quad \kappa = 0.20, \quad a = 0.60, \quad m = 0.50, \quad b = 0.80.$$

$$\text{From (2), } x^* = 0.625, \quad y^* \approx 1.7917.$$

The corresponding trace, determinant, and discriminant are

$\text{tr}(J^*) = -0.125$, $\det(J^*) = 0.5375$ and $D < 0$. Thus the positive equilibrium is a locally asymptotically stable focus.

Table 1. Effect of parameter variation on the feasibility margin.

Case	r	κ	m	b	$F = rb - \kappa m$	Regime
Nominal	1.2	0.2	0.5	0.8	0.86	Coexistence
Moderate uncertainty	1.08	0.22	0.55	0.72	0.6566	Coexistence
Near threshold	0.75	0.3	0.8	0.35	0.0225	Sensitive
Beyond threshold	0.7	0.32	0.85	0.34	-0.034	No positive equilibrium

The numerical results show that uncertainty does not merely shift the equilibrium; it can move the model close to the critical surface. In the near-threshold case, F remains positive but becomes small, so the equilibrium is highly sensitive to further parameter perturbations. When $F < 0$, the positive coexistence equilibrium disappears from the biologically admissible region.

8. Computational modeling procedure

For computational analysis, the following sequence is recommended: (i) specify nominal coefficients and uncertainty bounds; (ii) generate parameter realizations by a grid, Latin hypercube, or Monte Carlo sampling; (iii) compute the equilibrium for each realization; (iv) test feasibility; (v) evaluate the Jacobian eigenvalues, trace, determinant, discriminant, and $M(p)$ and (vi) classify the parameter point and visualize the result in a bifurcation or stability map. In higher-dimensional continuous models the criterion $\max \text{Re } \lambda(J^*) < 0$ is used, whereas for discrete-time models local stability requires all multipliers to lie inside the unit circle.

For fixed κ and b , the critical curve in the (r, m) -plane is $m = \frac{b}{\kappa}r$.

This curve provides a simple two-parameter visualization of uncertainty-induced transition between coexistence and non-coexistence regimes. For extended Lotka–Volterra models, analogous curves can be computed numerically from an eigenvalue condition.

9. Discussion

The analysis separates three effects of uncertainty: displacement of the equilibrium, variation of convergence rates and transient oscillations, and crossing of a critical parameter surface. This separation is useful because not every qualitative change should be called a bifurcation. In the selected continuous model, the principal critical event is loss of positive coexistence at $rb - \kappa m = 0$. Richer bifurcations can arise after adding delay, functional-response saturation, harvesting, seasonal forcing, or a discrete-time update rule [5–7].

The same uncertainty framework extends naturally to multispecies Lotka–Volterra systems. If $x^* = x^*(p)$ is a feasible equilibrium and $J^* = J(x^*, p)$, robust local stability requires $\max \operatorname{Re} \lambda(J^*) < 0$ for every $p \in U$ in continuous time. For a discrete model, the corresponding requirement is $\rho(J^*) < 1$. Hence the proposed procedure links uncertainty quantification directly with spectral stability analysis.

10. Conclusion

A mathematical modeling framework for stability and critical transitions in Lotka–Volterra systems with uncertain parameters has been developed. The parameters were represented by bounded perturbations, and their influence was propagated through the equilibrium and Jacobian characteristics. For the self-limited predator–prey model, explicit feasibility and stability conditions were obtained, and the critical surface $rb - \kappa m$ was identified. The numerical example demonstrated that uncertainty can substantially reduce the feasibility and stability margin. The study also established an important modeling limitation: the selected positive-parameter continuous model does not generate an interior Hopf bifurcation, so additional mechanisms are required when oscillatory bifurcation is the research target. The framework is suitable for further development toward discrete, delayed, stochastic, and multispecies Lotka–Volterra systems.

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