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MINIMAX PROBLEM FOR THE ROMANOVSKY DISTRIBUTION IN THE Lγ METRIC

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Abstract
The article solves the minimax problem for the Romanovsky distribution in the Lj metric. Convergence theorems for variation of the Romanovsky distribution to normal, binomial, negative-binomial distributions, Poisson distributions, Erlang distributions, and bet distributions are given.

Keywords: Convergence by variation, Romanovsky distribution, normal, binomial, negative-binomial, Poisson, Erlang and Bet distributions.

Introduction

This distribution is found in the work [1] of V. I. Romanovsky in connection with the following problem. Two ordered samples S_N, S_M of arbitrary volumes N and M are considered:

$$x_1 \le x_2 \le \ldots \le x_N$$
$$y_1 \le y_2 \le \ldots \le y_M \qquad (N \ge 1, \quad M \ge 1)$$

aus demselben stetigen Aggregat S mit Dichte *f(x),* was uns unbekannt ist. In der ersten Probe wird der Penis darin hervorgehoben x_{n+1}, Wir werden n Mitglieder haben, die nicht mehr als x_{n+1} und $N-n-1$ Angehörige von mindestens x_{n+1} In dieser Arbeit von V.I. Romanovsky wurde die Wahrscheinlichkeit festgestellt, dass in der zweiten Stichprobe μ nicht mehr als Mitglieder x_{n+1} und $M\text{-}\mu$ Mitgliedern mehr als x_{n+1}:

$$P_n(k) = P(\mu - k) = P_{N,M,n}(k) = P(k) =$$
$$= \begin{cases} \dfrac{C_{n+k}^{n} \quad C_{N+M-n-k-1}^{N-n-1}}{C_{N+M}^{N}}, & k = \overline{0,M} \\ 0, & k > M \end{cases}$$

$N \ge 1, \quad M \ge 1, \qquad n = \overline{0, N-1}$.

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The characteristic function of this distribution is also calculated there:

$$\varphi_n(t) = \frac{N!}{n!(N-n-1)!}\int_0^1 p_1^n q_1^{N-n-1}(p_1 e^{it} + q_1)^M dp_1$$

In here $q_1 = 1 - p_1$, and it is established that

$$M\mu = \frac{(n+1)M}{N+1},$$

$$D\mu = \frac{(n+1)(N-n)}{(N+1)(N+2)}\left(\frac{M^2}{N+1} + M\right).$$

V. I. Romanovsky proposes to use this distribution to construct a criterion for testing hypotheses about the homogeneity of the two samples under consideration. He gives a number of recurrence relations for probabilities $P_n(k)$.
V.I. Romanovsky studied this distribution in sufficient detail and pointed out its very important applied side. It is for this reason, and for the sake of brevity, that we will call this distribution the Romanovsky distribution.
This distribution has not been studied in terms of asymptotic properties.

Key results
Studying the asymptotic behavior of the P(k) distribution, we found that with various changes in parameters, it converges with the normal, binomial, Poisse, negative-binomial, Erlang, and beta distributions. Let

$$p = \frac{M}{N+M}, \quad q = \frac{N}{N+M}, \quad \alpha = \frac{n}{N}, \quad \beta = \frac{N-n}{N}$$

Next, let's introduce notations for distributions with which the Romanovsky distribution is similar.

$$T_1(k) = \frac{\sqrt{q}}{\sqrt{2\pi\alpha\beta p(N+M)}}\exp\left\{-\frac{u_k^2}{2}\right\},$$

$$u_k = \frac{kq - np}{\sqrt{2\pi\alpha\beta p(N+M)}}, \quad k = 0,1,2,...$$

- normal distribution.

$$T_2(k) = \frac{(n+k)!}{n!\,k!}p^{n+1}q^k, \quad k = 0,1,2,...,$$

$$T_3(k) = \frac{(k+N-n-1)!}{(N-n-1)!\,k!}p^{N-n}q^k, \quad k = 0,1,2,...$$

- negative binomial distributions,

$$T_4(k)=\begin{cases} C_M^k \alpha^k (1-\alpha)^{M-k}, & k=\overline{0,M} \\ 0, & k>M \end{cases}$$

- binomial distribution,

$$T_5(k)=\begin{cases} \dfrac{N!}{n!(N-n-1)!}\,\dfrac{1}{M}\left(\dfrac{k}{M}\right)^n\left(1-\dfrac{k}{M}\right)^{N-n-1}, & k=\overline{0,M} \\ 0, & k>M \end{cases}$$

- beta distribution,

$$T_6(k)=\frac{\left(\frac{np}{q}\right)^k}{k!}e^{-\frac{np}{q}}, \qquad k=0,1,2,....$$

$$T_7(k)=\frac{\left[\frac{(N-n)p}{q}\right]^k}{k!}\exp\left\{-\frac{(N-n)p}{q}\right\}, \qquad k=0,1,2,....$$

- Poisson distributions,

$$T_8(k)=\frac{\left(\frac{kq}{p}\right)^n\frac{q}{p}}{n!}\,e^{-\frac{kq}{p}}, \qquad k=0,1,2,....$$

$$T_9(k)=\frac{\left(\frac{kq}{p}\right)^{N-n-1}\frac{q}{p}}{(N-n-1)!}\,e^{-\frac{kq}{p}}, \qquad k=0,1,2,....$$

- Erlang distributions.

Let's also introduce the notations:

$$\rho(P,T_i,\gamma)=\left(\sum_{k=0}^{\infty}|R(k)-T_i(k)|^{\gamma}\right)^{1/\gamma}, \quad \gamma>0,\ i=1,2,\ldots,9$$

$$\chi_{N+M}(R,\gamma)=\sup_{0\le\alpha,p\le1}\ \min_{i=\overline{1,9}}\ \rho(R,T_i,\gamma).$$

The following theorems solve minimax problems of asymptotic behavior of the Romanovsky distribution. Theorem 1. Let $\gamma>0,5$. At $N+M\to\infty$

$$\chi_{N+M}(R,\gamma)=\Lambda_1(\gamma)(N+M)^{-\frac{2\gamma-1}{4\gamma}}+o\left((N+M)^{-\frac{2\gamma-1}{4\gamma}}\right),$$

Where is $\qquad \Lambda_1(\gamma)=\lambda_1^{\frac{\gamma+1}{3\gamma}}(\gamma)2^{-\frac{2\gamma-1}{3\gamma}},$

$$\lambda_1(\alpha,p,\gamma)=\frac{1}{6\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}|u^3(1+p)(1-2\alpha)-3u(q-2\alpha)|^{\gamma}e^{-\frac{\gamma u^2}{2}}du\right)^{\frac{1}{\gamma}}.$$

Let

$$D_1=\{(\alpha,p):\alpha<p<1-2\lambda_7(\gamma)\alpha\},D_2=\{(\alpha,p):1-\alpha<p<1-2\lambda_7(\gamma)(1-\alpha)\}$$

$$D_3=\{(\alpha,p):p<1-\alpha<1-\alpha\},\qquad D_4=\left\{(\alpha,p):\frac{1-p}{2\lambda_7(\gamma)}<\alpha<1-\frac{1-p}{2\lambda_7(\gamma)}\right\}.$$

Comparing the results of theorems 2 to 10, it can be shown that

$$\min_{i=\overline{1,9}}\rho(R,T_2,\gamma)=$$

$$=\begin{cases}\rho(R,T_2,\gamma),\quad если\quad (\alpha,p)\in D_1,\quad \alpha<\left(\frac{\lambda_1(\gamma)}{\lambda_2(\gamma)}\right)^{\frac{1}{2}}\times\\ \times(N+M)^{-1/4}+o\left((N+M)^{-1/4}\right),\\ \rho(R,T_3,\gamma),\quad если\quad (\alpha,p)\in D_2,\\ \qquad \alpha>1-\left(\frac{\lambda_1(\gamma)}{\lambda_2(\gamma)}\right)^{1/2}\times\\ \times\quad (N+M)^{-1/4}+o\left((N+M)^{-1/4}\right)\\ \rho(R,T_4,\gamma),\quad если\quad (\alpha,p)\in D_3,\quad p<\left(\frac{\lambda_1(\gamma)}{\lambda_2(\gamma)}\right)^{1/2}\times\\ \qquad\times\ (N+M)^{-1/4}+o\left((N+M)^{-1/4}\right),\\ \rho(R,T_5,\gamma),\quad если\quad (\alpha,p)\in D_4,\quad p>1-\left(\frac{\lambda_1(\gamma)}{\lambda_2(\gamma)}\right)^{1/2}\times\\ \quad\times\frac{1}{\sqrt[4]{2\lambda_2(\gamma)}}(N+M)^{-1/4}+o\left((N+M)^{-1/4}\right)\\ \rho(R,T_1,\gamma),\quad в\ \ остальных\quad случаях.\end{cases}$$

Let $D_5=\{(\alpha,p):\quad \alpha<p<1-\lambda_3\alpha\},D_6=\{(\alpha,p):\quad 1-\alpha<p<1-\lambda_3(1-\alpha)\},$

$$D_7=\{(\alpha,p):\quad p<\alpha<1-p\},\quad D_8=\left\{(\alpha,p):\quad \frac{1-p}{2\lambda_3}<\alpha<1-\frac{1-p}{2\lambda_3}\right\}$$

From the theorems it is easy to deduce that

$$\min_{i=\overline{1,9}}\rho(R,T_i)=$$

$$
=\begin{cases}
r(R,T_2), \quad \textit{если} \quad (\alpha,p)\in D_5, \quad \alpha<\sqrt{\dfrac{\lambda_4}{\lambda_3}}(N+M)^{-1/4}+ \\
\qquad +o\left((N+M)^{-1/4}\right), \\
r(R,T_3), \quad \textit{если} \quad (\alpha,p)\in D_6, \quad \alpha<1-\sqrt{\dfrac{\lambda_4}{\lambda_3}}(N+M)^{-1/4}+ \\
\qquad +o\left((N+M)^{-1/4}\right) \\
r(R,T_4), \quad \textit{если} \quad (\alpha,p)\in D_7, \quad p<\sqrt{\dfrac{\lambda_4}{\lambda_3}}(N+M)^{-1/4}+ \\
\qquad +o\left((N+M)^{-1/4}\right), \\
r(R,T_5), \quad \textit{если} \quad (\alpha,p)\in D_8, \quad p>\sqrt{2\lambda_4}(N+M)^{-1/4}+ \\
\qquad +o\left((N+M)^{-1/4}\right) \\
r(R,T_1), \quad \textit{в остальных случаях}
\end{cases}
$$

Theorem 3. Let $\gamma>0{,}5$. At $\sigma\to\infty$

$$\rho(R,T_1,\gamma)=\frac{\lambda_1(\alpha,p,\gamma)}{(\sigma q)^{\frac{2\gamma-1}{\gamma}}}+O\left((\sigma q)^{-\frac{3\gamma-1}{\gamma}}\right),$$

Theorem 4. At $\alpha=o\left((N+M)^{-\frac{1-\gamma}{1+\gamma}}\right),\ o<\gamma<1, \alpha=o(n^{-\varepsilon}),\ \gamma\ge 1$

$$\rho(R,T_2,\gamma)=\frac{\lambda_2(\gamma)\alpha}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+\frac{\alpha}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}O\left(\min\left(1,\frac{1}{\sigma q}\right)\right).$$

Theorem 5. At $\beta=o\left((N+M)^{-\frac{1-\gamma}{1+\gamma}}\right)\ o<\gamma<1\ \beta=o(n^{-\varepsilon}),\ \gamma\ge 1$

$$\rho(R,T_3,\gamma)=\frac{\lambda_2(\gamma)\beta}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+\frac{\beta}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}\cdot O\left(\min\left(1,\frac{1}{\sigma q}\right)\right)$$

Theorem 6. At $\alpha=o\left((N+M)^{-\frac{1-\gamma}{1+\gamma}}\right),\ o<\gamma<1, \alpha=o(n^{-\varepsilon}),\ \gamma\ge 1$

$$\rho(R,T_4,\gamma)=\frac{\lambda_2(\gamma)p}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+\frac{p}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}\cdot O\left(\min\left(1,\frac{1}{\sigma q}\right)\right).$$

Theorem 7. At $q=o\left((N+M)^{-\frac{1-\gamma}{1+\gamma}}\right)$ $o<\gamma<1$ $q=o(n^{-\varepsilon}),\ \gamma\geq 1$

$$\rho(R,T_5,\gamma)=\frac{q}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+O\left(\min\left(\frac{q}{(\sigma q)^{\frac{2\gamma-1}{\gamma}}}\right)\right).$$

Theorem 8. At $\alpha,p=o\left((N+M)^{-\frac{1-\gamma}{1+\gamma}}\right),o<\gamma<1$, $\alpha,p=o(n^{-\varepsilon}),\ \gamma\geq 1$

$$\rho(R,T_6,\gamma)=\frac{(\alpha+p)\lambda_3(\gamma)}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+\frac{\alpha+p}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}\cdot O\left(\min\left(1,\frac{1}{\sigma q}\right)\right).$$

Theorem 9. At $\beta,q=o\left((N+M)^{-\frac{1-\gamma}{1+\gamma}}\right)$ $o<\gamma<1$ $\beta,q=o(n^{-\varepsilon}),\ \gamma\geq 1$

$$\rho(R,T_7,\gamma)=\frac{(\beta+q)\lambda_3(\gamma)}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+\frac{\beta+q}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}O\left(\min\left(1,\frac{1}{\sigma q}\right)\right).$$

Theorem 10. At $\gamma\geq 1,\ \alpha,q<\frac{1}{N}$ or
$\alpha,q=o((N+M)^{-(1-\gamma)/(1+\gamma)}),\ \ 0{,}5<\gamma<1$

$$\rho(R,T_8,\gamma)=\frac{(\alpha+q)\lambda_3(\gamma)}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+(\alpha+q)O\left((\sigma q)^{-\frac{2\gamma-1}{\gamma}}\right).$$

Theorem 11. At $\gamma\geq 1,\ \beta,q<\frac{1}{N}$ or

$$\beta,q=o((N+M)^{-(1-\gamma)/(1+\gamma)}),\ \ 0{,}5<\gamma<1$$

$$\rho(R,T_9,\gamma)=\frac{(\beta+q)\lambda_3(\gamma)}{(\sigma q)^{\frac{\gamma-1}{\gamma}}}+(\beta+q)\cdot O\left((\sigma q)^{-\frac{2\gamma-1}{\gamma}}\right).$$

Beweis des Satzes 2. Zuerst werden wir einen Beweis des Satzes in dem Fall geben, in dem $\gamma=1$. Summen, bei denen der Summenindex zu Ungleichungen erfüllt $|u_k|<\sigma^{1/3},\ \ |u_k|\geq\sigma^{1/3}$ Lassen Sie uns entsprechend bezeichnen. Betrachten wir zuerst

$$\Sigma_2\,|\,P(k)-T_1(k)\,|.$$

It follows from lemma 2.1[3] that

$$\sum_{|u_k|\ge\sigma^{1/3}} T_1(k) = O\left(\frac{1}{\sigma^2}\right). \quad (1)$$

$$\Sigma_2 R(k) \le \frac{1}{\sigma^4}\sum_{k=0}^{\infty}\left(k-\frac{np}{q}\right)^3 R(k) = \frac{1}{\sigma^4}\left(np\left(1+O\left(\frac{1}{N}\right)+O\left(\frac{1}{M}\right)\right)\right) = O\left(\frac{1}{\sigma^2}\right) \quad (2)$$

From (1), (2) it follows that

$$\Sigma_2 |R(k)-T_1(k)| = O\left(\frac{1}{\sigma^2}\right). \quad (3)$$

Now consider those values of k for which $|u_k| < \sigma^{\frac{1}{3}}$.

Given the limitations of you and keeping in mind $N\to\infty$, $M\to\infty$, $n\to\infty$, $\sigma\to\infty$ получим

$$N-n=\beta N\to\infty,\ M-k=\beta N\left(1-\frac{u_k\alpha}{\sigma}\right)\to\infty,\ k=\frac{np}{q}\left(1+\frac{u_k\beta}{\sigma}\right)\to\infty,$$

then for factorials R(k) the Stirling formula is applicable:

$$m! = \sqrt{2\pi m}\cdot m^m\cdot e^{-m}\left(1+\frac{1}{12m}+O\left(\frac{1}{m^2}\right)\right) \quad (4)$$

Simple calculations show that

$$R(k) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{u_k^2}{2}}\left[1+\frac{u_k^3(1+p)(1-2\alpha)}{6\sigma q}-\frac{u_k(q-2\alpha)}{2\sigma q}+O\left(\frac{P_3(u_k)}{(\sigma q)^2}\right)\right],$$

(5)

Where is

$P_3(k) = (u_k(1+p)\cdot(1-2\alpha)-3u(q-2\alpha))^2 + \alpha p u_k^4 + u_k^6(1+p)(\alpha^4+\beta^4)$.

Since

$$\Sigma_8 |P_3(u_k)|\, e^{-\frac{u_k^2}{2}}\Delta u_k = O(1) \quad (6)$$

then it follows from (3), (5), (6) that

$$\rho(R,T_1) = \frac{1}{6\sqrt{2\pi}}\Sigma_8 |u_k^3+(1+p)(1-2\alpha)-3u_k(q-2\alpha)|\, e^{-\frac{u_k^2}{2}}\cdot\Delta u_k + O\left(\frac{1}{\sigma^2}\right) \quad (7)$$

Unter Verwendung von Lemma 1 [5] bis (7) erhalten wir einen Beweis für Satz 2, im Falle von $\gamma=1$ Betrachten wir nun den Fall, in dem $\gamma>1$. Lassen

$$J_3 = \frac{1}{6\sigma}\left(\sum_{k=0}^{\infty}|u_k^3(1+p)(1-2\alpha)-3u_k(q-2\alpha)|^{\gamma}\cdot T_2^{\gamma}(k)\right)^{1/\gamma},$$

$$J_4=\left(\sum_{k=0}^{\infty}R(k)-T_1-\frac{1}{6\sigma}(u_k^3(1+p)(1-2\alpha)-3u_k(q-2\alpha)-3u_k(q-2)T_1(k)|\right)^{1/\gamma}.$$

Using lemma 2 [5] we have $J_3-J_4\le\rho(R,T_1,\gamma)\le J_3+J_4$ (8)

Let's consider first J_4

$$J_4^{\gamma}=\sum_{|u_k|<\sigma^{1/3}}+\sum_{|u_k|\ge\sigma^{1/3}}|R(k)-T_1(k)-$$

$$-\frac{1}{6\sigma}u_k^3(1+p)(1-2\alpha)-3u_k(q-2\alpha)\cdot T_1(k)|^{\gamma}=J_{41}+J_{42},$$

$$J_{42}\le 3^{\gamma}\left(\sum_{|u_k|\ge\sigma^{1/3}}R^{\gamma}(k)+\sum_{|u_k|\ge\sigma^{1/3}}T_1^{\gamma}(k)+\right.$$

$$\left.+\left(\frac{1}{6\sqrt{2\pi}\,\sigma^2}\right)^{\gamma}\sum_{|u_k|\ge\sigma^{1/3}}|u_k^3(1+p)(1-2\alpha)-3u_k(q-2\alpha)|^{\gamma}\,e^{-\frac{\gamma u_k^2}{2}}\right).$$

Similarly, it can be obtained that

$$\sum_{|u_k|\ge\sigma^{1/3}}R^{\gamma}(k)<\sum_{|u_k|\ge\sigma^{1/3}}R(k)<2\exp\left\{-\frac{\beta\sigma^{2/3}}{4}\right\}=O(\sigma^{-3\gamma}) \qquad (9)$$

Using lemma 1[5], it can be shown that

$$\frac{1}{(\sqrt{2\pi})^{\gamma}\sigma^{\gamma-1}}\sum_{|u_k|\ge\sigma^{1/3}}e^{-\frac{\gamma u^2}{2}}\Delta u_k=O\left(\frac{1}{\sigma^{3\gamma-1}}\right).$$

(10)

And

$$\frac{1}{(6\sqrt{2\pi})^{\gamma}\sigma^{2\gamma-1}}\sum_{|u_k|\ge\sigma^{1/3}}|(1+p)(1-2\alpha)u_k^3-3u_k(1-2\alpha)|^{\gamma}\,e^{-\frac{\gamma u_k^2}{2}}\Delta u_k=O\left(\frac{1}{\sigma^{3\gamma-1}}\right) \qquad (11)$$

From (9)-(11) it follows that

$$J_{42}=O\left(\frac{1}{\sigma^{3\gamma-1}}\right)$$

(12)

From (5) it follows that

$$J_{41}=\sum_{|u_k|<\sigma^{1/3}}\left|R(k)-T_1(k)-\frac{1}{6\sigma}u_k^3(1+o)(1-2\alpha)-3u(q-2\alpha)T_1(k)\right|^{\gamma}=$$

$$= O\left(\frac{1}{\sigma^{3\gamma-1}}\sum_{|u_k|<\sigma^{1/3}}|P_3(u)|^{\gamma}e^{-\frac{\gamma u_k^2}{2}}\Delta u_k\right).$$

Since
$$\sum_{|u_k|<\sigma^{1/3}}|P_3(u_k)|^{\gamma}e^{-\frac{\gamma u^2}{2}}\Delta u_k = O(1),$$

that
$$J_{41} = O\left(\frac{1}{\sigma^{3\gamma-1}}\right).$$

(13)

Using the lemma 2.1 [3] к J_3 Have

$$J_3 = \frac{1}{6\sqrt{2\pi}\,\sigma^{\frac{2\gamma-1}{\gamma}}}\int_{-\infty}^{\infty}|(1+p)(1-2\alpha)u^3 - 3u(q-2\alpha)|^{\gamma}\times e^{-\frac{u^2}{2}}du + O\left(\frac{1}{\sigma^{\frac{3\gamma-1}{\gamma}}}\right).$$

(14)

From (12)-(14) follows the validity of the theorem in the case when $\gamma \geq 1$

Fall $0,5<\gamma<1$ wird durch die Gelder-Ungleichung in der gleichen Weise bewiesen wie der obige Teil des Beweises des Satzes .

Proof of Theorem 3. First, we will give a proof of the theorem in the case when $\gamma = 1$.

Let

$$z_k = k - \frac{np}{q}, \qquad \eta_6 = \max\left(\alpha^2, \frac{\alpha}{N+M}\right)$$

and
$$J_5 = \sum_{|z_k|<\frac{1}{2}\sqrt{N+M}}|R(k)-T_2(k)| = \Sigma_3|R(k)-T_2(k)|,$$

$$J_6 = \sum_{|z_k|\geq\frac{1}{2}\sqrt{N+M}}|R(k)-T_2(k)| = \Sigma_4|R(k)-T_2(k)|.$$

Simple calculations show that

$$\sum_{k=0}^{\infty}z_k^4R(k) = \left[\frac{6n^2p^2}{q^2}\left(\frac{2p^2}{q^2}+\frac{p}{q}+1\right)+\frac{np}{q}\left(\frac{6p^3}{q^3}+\frac{24p^2}{q^2}+\frac{21p}{q}+1\right)+\right.$$
$$\left.+\frac{p}{q}\left(\frac{9p^3}{q^3}+\frac{18p^2}{q^2}+\frac{15p}{q}+1\right)\right]\left(1+O\left(\frac{1}{N+M}\right)\right)$$

(16)

and $$\sum_{k=0}^{\infty} z_k^4 T_2(k) = \frac{3(n+1)^2 \cdot p^2}{q^4} + \frac{6(n+1)p^2}{q^4} + \frac{(n+1)p}{q^2}$$ (17)

From (16), (17) it follows that

$$J_6 = O(\eta_6)$$ (18)

Now consider those values of k for which

$$|z_k| < \frac{1}{2}\sqrt{N+M}$$

$$J_5 = \Sigma_4 |l_2(k) - 1| T_2(k),$$

Where is $$l_2(k) = \frac{R(k)}{T_2(k)}.$$

Simple calculations show that

$$l_2(k) = 1 + \frac{\alpha}{2}(1-u_k^2) + \frac{u_k \alpha q}{2\sigma} + O\left(\frac{\alpha^2}{\sigma^2}\right). \quad (19)$$

Due to the well-known inequalities between the moments of

$$\sum_{k=0}^{\infty} |z_k| T_2(k) < \left(\sum_{k=0}^{\infty} z_k^2 T_2(k)\right)^{\frac{1}{2}} = \left(\frac{(n+1)p}{q^2}\right)^{\frac{1}{2}} < \frac{(n+1)p}{q^2}$$

At $\frac{(n+1)p}{q^2} < 1$

$$\Sigma |z_k| T_2(k) = \frac{(n+1)p}{q} T_2(o) + \left(1 - \frac{(n+1)p}{q}\right) T_2(1) + \sum_{|z_k|>1} |z_k| T_2(k) < \frac{3(n+1)p}{q^2},$$

$$\Sigma |z_k| T_2(k) < 3\frac{(n+1)p}{q^2} .$$ (20)

So $$\rho(R,T_2) = \frac{\alpha}{2}\Sigma_{10} |1-u_k^2| T_2(k) + O(\eta_6). \quad (21)$$

Further from (17)-(21) meaning $\eta_6 = o(\alpha)$ Get

$$\rho(R,T_2) = O(\alpha).$$

Using now to estimate the probability of $T_2(k)$ asymptotic decomposition and applying lemma 1 [5] to expression (21), it can be shown that when $(n+1)p \to \infty$

$$\sum_{k=0}^{\infty}|1-u_k^2|T_2(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|1-u^2|e^{-u^2/2}du+O\left(\frac{q}{\sqrt{(n+1)p}}\right)$$

(22)

From (21), (22) follows the validity of the theorem in the case of $\gamma=1$.

Proof of a theorem for $\gamma>0,5$ Based on chance $\gamma=1$ and is conducted similarly to the proof of theorem 2, so we will not cite it.

The proof of Theorem 4 is analogous to the proof of Theorem 3 for a random variable $\mu'=M-\mu$.

Proof of Theorem 5 in the case of $\gamma=1$. Let

$$\eta_7=\max\left(p^2,\frac{p}{N+M}\right),z_k=k-\alpha M\,,$$

$$J_7=\sum_{|z_k|<\frac{1}{2}\sqrt{N+M}}|R(k)-T_4(k)|=\Sigma_{21}|R(k)-T_4(k)|,$$

$$J_8=\sum_{|z_k|\geq\frac{1}{2}\sqrt{N+M}}|R(k)-T_4(k)|=\Sigma_{12}|R(k)-T_4(k)|$$

Since

$$\sum_{k=0}^{\infty}z_k^4c_M^k\alpha^k(1-\alpha)^{M-k}=3\alpha^2\beta^2M^2+\alpha\beta M(1-6\alpha\beta)=(N+M)^2O(\eta_7)$$

(23)

и

$$\sum_{k=0}^{\infty}z_k^4\,R(k)=$$

$$=[3\alpha^2\beta^2M^2+\alpha M(1-7\alpha+12\alpha^2-6\alpha^2)]\left(1+O\left(\frac{1}{N}\right)\right)=(N+M)^2O(\eta_7) \qquad (24)$$

it follows from (23), (24) that

$$J_8=O(\eta_7)$$

(25)

Let us now consider those values of k for which

$$|z_k|<\frac{1}{2}\sqrt{N+M}\,,\quad \text{т.е.}\quad J_7=\Sigma_{12}|l_4(k)-1|\cdot T_4(k),$$

Where is

$$l_4(k)=\frac{R(k)}{T_4(k)}=\frac{N!(N+M-n-k-1)!(k+n)!}{(N+M)!(N-n-1)!\,n!}\cdot\alpha^{-k}(1-\alpha)^{-M+k}.$$

It follows from the condition of the theorem that
$N \to \infty$, that

$$N-n-1=\beta N\left(1-\frac{1}{\beta N}\right)\to\infty,\ n=\alpha N\to\infty.$$

Also, für alle Fakultäten $l_4(k)$ die Stirling-Formel (4) anwendbar ist. Einfache Berechnungen zeigen, dass

$$l_4(k)=1-\frac{p}{2}\left(1-\frac{z_k^2}{\alpha\beta M}\right)+\frac{z_k p}{2\alpha\beta M}+O\left(\frac{p^2 P_4(z_k)}{(\alpha\beta M)^{3/2}}\right), \quad (26)$$

Where is $P_4(z_k)=z_k^2(\alpha+\beta\rho)+z_k^3(\alpha-\beta)$.

From (26) we get $J_7=\frac{p}{2}\Sigma_{11}\left|1-\frac{z_k^2}{\alpha\beta M}+\frac{z_k}{\alpha\beta M}\right|T_4(k)+R_4$.

Since

$$\Sigma_{11}z_k^2T_4(k)<\alpha\beta M=(N+M)^2O(\eta_7),$$

$$\Sigma_{11}|z_k|^3\,T_4(k)\le(\Sigma_{11}z_k^4T_4(k))^{3/4}=(N+M)^2O(\eta_7),$$

that $R_4=O(\eta_7)$. (27)

from (25)-(27) it follows that

$$\rho(R,T_4)=O(p).$$

(28)

Due to the well-known moments between the moments

$$\Sigma|z_k|c_M\alpha^k\beta^{M-k}<(\Sigma z_k^2c_M^k\alpha^k\beta^{M-k})^{1/2}=(\alpha\beta M)^{1/2}<\alpha\beta M,$$

at $\alpha\beta M\ge 1$. (29)

And when $\alpha\beta M<1$

$$\Sigma|z_k|T_4(k)=\alpha MT_4(o)+(1-\alpha M)T_4(1)+\sum_{|z_k|>1}|z_k|T_4(k)<3\alpha M \quad .$$

(30)

It follows from (26)-(30) that

$$J_7=\frac{p}{2}\Sigma_{11}|1-u_k^2|T_4(k)+O(\eta_7).$$

(31)

Applying lemma 1 [5] to (29), we obtain a proof of theorem 5 in the case of $\gamma=1$.

Proof of Theorem 6. Limiting commonalities can be considered $n\ge 2,\ \ N-n\ge 3,\ \ N<M^{2/3}$.

Let

$$\rho(R,T_5)=J_9+J_{10}+J_{11},$$

Where is $J_9 = \sum_{k<\sqrt{M}} |R(k)-T_5(k)|, \quad J_{10} = \sum_{\sqrt{M} \le k < M-\sqrt{M}} |R(k)-T_5(k)|$

and $$J_{11} = \sum_{k \ge M-\sqrt{M}} |R(k)-T_5(k)|.$$

Let's evaluate first J_9. Ratio

$$\sum_{k<\sqrt{M}} R(k) \le \frac{N!}{n!(N-n-1)} \times \sum_{k<\sqrt{M}} \left(\frac{k}{M}\right)^n \left(1-\frac{k}{M}\right)^{N-n-1} \cdot \frac{1}{M} = O\left(M^{-\frac{n+1}{2}}\right) \quad , \tag{32}$$

$$\sum_{k<\sqrt{M}} T_5(k) \le \sqrt{M} \frac{N!}{n!(N-n-1)!} \cdot \left(\frac{1}{\sqrt{M}}\right)^n \cdot \frac{1}{M} = O\left(M^{-\frac{n+1}{2}}\right) \tag{33}$$

shows that $$\sum_{k>M-\sqrt{M}} R(k) \le \sum_{M-k<\sqrt{M}} R(k) = O\left(M^{-\frac{N-n}{2}}\right). \tag{34}$$

It is not difficult to show that

$$J_9 = O\left(M^{-\frac{n+1}{2}}\right) \tag{35}$$

and $$\sum_{k>M-\sqrt{M}} T_5(k) = O\left(M^{-\frac{N-n}{2}}\right). \tag{36}$$

From (35), (36) it follows that

$$J_{11} = O\left(M^{-\frac{N-n}{2}}\right).$$

Since $N-n-1 \ge 2, \quad n \ge 2$, то

$$J_9 = O\left(M^{-3/2}\right), \quad J_{11} = O\left(M^{-3/2}\right). \tag{37}$$

Now let's consider J_{10}. It is not difficult to show that

$$R(k) = \frac{N!}{n!(N-n-1)!} \cdot \frac{(M-k)^{N-n-1} k^n}{M^n} \times$$
$$\times \prod_{i=1}^{n} \left(1+\frac{i}{k}\right) \prod_{i=1}^{N-n-1} \left(1+\frac{i}{M-k}\right) \Big/ \prod_{i=1}^{n} \left(1+\frac{i}{M}\right).$$

Simple calculations show that

$$R(k) = T_5(k)\left[1+\frac{n(n+1)}{2k}+\frac{(N-n-1)(N-n)}{2(M-k)}-\right.$$

$$-\frac{N(N+1)}{2M}+O\left(\frac{n^3}{k^3}\right)+O\left(\frac{N^3}{(M-k)^2}\right)+O\left(\frac{N^3}{M^2}\right)\Bigg]$$

(38)

From (38) it can be shown that

$$J_{10}=\frac{N^2(N-1)(N+1)}{2n(N-n-1)M}\times\sum_{\sqrt{M}\le k<M-\sqrt{M}}\left|\left(\frac{k}{M}\right)^2-\frac{2(n+1)k}{(N+1)M}+\frac{n(n+1)}{N(N+1)}\right|\times$$

$$\times\frac{(N-2)!}{(N-n-2)!(n-1)!}\cdot\left(\frac{k}{M}\right)^{n-1}\left(1-\frac{k}{M}\right)^{N-n-2}\frac{1}{M}++O\left(\frac{N^2}{M^2}\Sigma_{10}P_5\left(\frac{k}{M}\right)\right),(39)$$

Where is

$$P_5\left(\frac{k}{M}\right)=\left[\left(\frac{k}{M}\right)^3+\frac{n(n+1)\cdot(2n+1)}{12N^2(N+1)}\right]\times$$

$$\times\frac{(N-3)!}{(N-n-3)!(n-2)!}\left(\frac{k}{M}\right)^{n-2}\left(1-\frac{k}{M}\right)^{N-n-3}.$$

Applying Lemma 2.1[5] Prokhorov Y.V. to (39) we have

$$J_{10}=\frac{q}{2}\int_0^1\left|x^2-2\frac{n+1}{N+1}x+\frac{n(n+1)}{N(N+1)}\right|\beta(x)dx+$$

$$+O\left(q^{3/2}\int_0^1\left(x^3+\frac{n(n+1)(2n+1)}{12N^2(N+1)}\right)\beta'(x)dx\right),\qquad(40)$$

Where is $x=\frac{k}{M}$,

$$\beta(x)=\frac{(N-2)!}{(N-n-2)!(n-1)!}x^{n-1}\cdot(1-x)^{N-n-2}\cdot\frac{1}{M},$$

$$\beta'(x)=\frac{(N-4)!}{(N-n-3)!(n-2)!}x^{n-2}\cdot(1-x)^{N-n-3}\cdot\frac{1}{M}.$$

At $N\to\infty$ beta distribution tends to normal. Taking this into account, from (40) (meaning (37) we obtain a proof of the theorem for $\gamma=1$.

Now consider the case $\gamma>1$. Let us first consider those values of k for which $k<\sqrt{M}$

$$J_{12}=\sum_{k<\sqrt{M}}|R(k)-T_5(k)|^{\gamma}\le 2^{\gamma}\left(\sum_{k<\sqrt{M}}R^{\gamma}(k)+\sum_{k<\sqrt{M}}T_5^{\gamma}(k)\right).$$

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It follows from the lemma that

$$\sum_{k<\sqrt{M}} T_5^{\gamma}(k) \le \sum_{k<\sqrt{M}} T_5(k) < \exp\left\{-\frac{M}{2}\left(1-\frac{1}{2\alpha\beta}\right)\right\}$$

and
$$\sum_{k<\sqrt{M}} R^{\gamma}(k) \le \sum_{k<\sqrt{M}} R(k) < \exp\left\{-\frac{M}{2}\left(1-\frac{qp}{2\alpha\beta}\right)\right\},$$

that
$$J_{12} = \sum_{k<\sqrt{M}} |R(k)-T_5(k)|^{\gamma} = O\left(M^{-\frac{3\gamma}{2}}\right)$$

(41)

Similarly (41) it may be shown that

$$J_{14} = \sum_{k\ge M-\sqrt{M}} |R(k)-T_5(k)|^{\gamma} = O\left(M^{-\frac{3\gamma}{2}}\right)$$

(42)

Now consider those values of k for which

$\sqrt{M} \le k < M-\sqrt{M}$ From (40) it follows that

$$J_{13} = \left(\frac{q}{2}\right)^{\gamma} \sigma \int_0^1 \left| x^2 - 2\frac{n+1}{N+1}x + \frac{n(n+1)}{N(N+1)} \right|^{\gamma} \beta^{\gamma}(x)dx + O\left(\frac{q^{3\gamma/2}}{\sigma}\right) \quad (43)$$

Since $J_{12} = o(J_{13})$ и $J_{14} = o(J_{13})$, then from (41)-(43) follows the proof of the theorem in the case of $\gamma > 1$. Proof of a theorem in the case of $0,5 < \gamma < 1$ is proved with the help of the Gelder inequality.

Proof of Theorem 7. Let's consider first $\gamma = 1$. Let

$$z_k = k - \frac{np}{q}, \qquad \eta_8 = \max\left((\alpha+p)^2, \frac{\alpha+p}{N+M}\right)$$

It can be shown that

$$\sum_{k=0}^{\infty} z_k^4 T_6(k) = 3\left(\frac{np}{q}\right)^2 + \frac{np}{q}.$$

(44)

From (16), (44) it follows that

$$\sum_{|z_k|\ge\frac{1}{2}\sqrt{N+M}} |R(k)-T_6(k)| = O(\eta_8)$$

(45)

Now consider those values of k for which

$|z_k| < \frac{1}{2}\sqrt{N+M}$

$$\sum_{|z_k|<\frac{1}{2}\sqrt{N+M}} |R(k)-T_6(k)| = \sum_{|z_k|<\frac{1}{2}\sqrt{N+M}} |l_6(k)-1|T_6(k),$$

Where is

$$l_6(k)=\frac{R(k)}{T_6(k)}=\frac{N!M!(N+M-n-k-1)!(k+n)!}{(N+M)!(N-n-1)!(M-k)!n!}\cdot\left(\frac{np}{q}\right)^{-k} e^{\frac{np}{q}}.$$

Außerhalb der Begrenzung z_k in dieser Zone und von den Bedingungen $N\to\infty$, $M\to\infty, n\to\infty$ Daraus folgt, dass

$$N-n-1=\beta N\to\infty,\quad M-k=\beta M\left(1-\frac{z_k}{\beta M}\right)\to\infty.$$

Then for all factorials $l_6(k)$ the Stirling formula (4) is applicable. It is not difficult to show that

$$l_6(k)=1+\frac{\alpha+p}{2}\left(1-\frac{z_k^2 p}{np}\right)+\frac{(\alpha+p)z_k q}{2np}+O\left(\alpha^2+p^2+\frac{z_k^2+z_k^4}{(np)^2}\right). \quad (47)$$

Since

$$\sum_{|z_k|<\frac{1}{2}\sqrt{N+M}} |z_k|T_6(k)\le\left(\sum_{k=0}^{\infty} z_k^2 T_6(k)\right)^{1/2}=\frac{np}{q}, \quad (48)$$

at $\frac{np}{q}\ge 1$.

For values $\frac{np}{q}<1$ Have

$$\sum_{|z_k|<\frac{1}{2}\sqrt{N+M}} |z_k|T_6(k)=\frac{np}{q}T_6(o)+\left(1-\frac{np}{q}\right)T_6(1)+\sum_{|z_k|>1}|z_k|T_6(k)\le\frac{3np}{q}.$$

(49)

From (47)-(49) it follows that

$$\sum_{k=0}^{\infty}|R(k)-T_6(k)|=\frac{\alpha+p}{2}\sum_{|z_k|<\frac{1}{2}\sqrt{N+M}}\left|1-\frac{z_k^2 q}{np}\right|T_6(k)+O(\eta_8)+pO\left(\frac{q}{\sqrt{np}}\right).$$

(50)

Verwendung zur Evaluierung $T_6(k)$ Lemmu 1 [],

Es kann gezeigt werden, dass, wenn

$\frac{np}{q} \to \infty$

$$\sum_{|z_k|<\frac{1}{2}\sqrt{N+M}} \left| 1-\left(\frac{x_k\sqrt{q}}{\sqrt{np}}\right)^2 \right| T_6(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |1-u^2| e^{-\frac{u^2}{2}} du + O\left(\frac{\sqrt{q}}{\sqrt{np}}\right).$$

(51)

From (45), (50), (51) follows the validity of the theorem in the case of $\gamma=1$. Cases $\gamma>1$, $0{,}5<\gamma<1$ are conducted similarly to the proof of theorem 2.

The proof of Theorem 8 is similar to the proof of Theorem 7.

Proof of Theorem 9. Let

$$z'_k = k - \frac{n+1}{q}.$$

In [4], the characteristic function of the Erlang distribution is given:

$$\varphi(t) = \left(1-\frac{it}{q}\right)^{-(n+1)}.$$

Using $\varphi(t)$ Have

$$\sum_{k=0}^{\infty}(z'_k)T_8(k) = n^4 + 6(1-2p)n^3 + (5+8p)n^2 + (6+p^2)n$$

(52)

From (16), (52) and from the theorem condition $q<\frac{1}{N}$ It can be shown that

$$\sum_{|z'_k|\ge\frac{1}{2}\sqrt{N+M}} |R(k)-T_8(k)| = O(\alpha^2).$$

(53)

Now consider those values of k for which

$|z'_k| \ge \frac{1}{2}\sqrt{N+M}$. Simple calculations show that

$$R(k)=\frac{\left(\frac{kp}{p}\right)^n \frac{q}{p}}{n!}e^{-\frac{kq}{p}}\cdot\prod_{i=1}^{n}\left(1-\frac{i}{N}\right)\times$$

$$\prod_{i=1}^{N-n-1}\left(1+\frac{i}{M-k}\right)\prod_{i=1}^{n}\left(1+\frac{i}{k}\right)\Big/\prod_{i=1}^{N}\left(1+\frac{1}{M}\right)e^{-\frac{kq}{p}}=$$

$$+\sum_{i=1}^{N-n-1}\ln\left(1-\frac{i}{M-k}\right)+\sum_{i=1}^{n}\ln\left(1+\frac{i}{n}\right)-\sum_{i=1}^{N}\ln\left(1+\frac{i}{M}\right)\Bigg\}.$$

It can be shown that

$$R(k)=T_8(k)\Bigg[1+\frac{\alpha+q}{2}\left(1-\frac{(z'_k)^2q^2}{n+1}\right)+$$

$$-\frac{(z'_k)^3+z'_k}{n+1}q^2+\frac{(n+1)q^2}{2}-\frac{\alpha^2 n}{2}+\frac{\alpha}{N}+\frac{q}{12}\alpha+$$

$$+nO(\alpha^3+q^3)+O\left(\frac{q(z'_k)^4+\alpha\beta(z'_k)^2}{(n+1)^{3/2}}q^{3/2}\right)\Bigg] \tag{54}$$

It follows from (54) that

$$\sum_{|z'_k|<\frac{1}{2}\sqrt{N+M}}|R(k)-T_8(k)|=\frac{\alpha+q}{2}\sum_{|z'_k|<\frac{1}{2}\sqrt{N+M}}\left|1-\frac{(z'_k)^2q^2}{n+1}\right|T_8(k)+R_8 \tag{55}$$

Out of Limitation z'_k, from the conditions $\sigma\to\infty,\ \alpha,q<\frac{1}{N}$ It follows that

$$R_8=O(\alpha^2+q^2)$$

(56)

From (53)-(58) it follows that

$$\rho(R,T_8)=O(\alpha+q)$$

(57)

Further, when $n\to\infty$ the distribution of Erlang tends towards normal law. Taking this into account, and from (55) we obtain a proof of theorem 9.

The proof of theorem 10 is analogous to proof 9 for a random quantity $\mu'=M-\mu$.

Proof of Theorem 1 in the case of $\gamma=1$. Studying the asymptotic behavior of the Romanovsky distribution in the case when $\sigma<\infty$ came to the following conclusion: if $\sigma<\infty$

,

then the Romanovsky distribution is close to binomial, negative-binomial, Poisson distributions. From the condition $\alpha p = O\left(\frac{1}{N+M}\right)$ It follows that

$$\min_{i=2,3,4,6,7} \rho(R,T_i) =$$

$$= \begin{cases} \rho(R,T_1), & если \quad \alpha < p, \alpha < \frac{1}{\sqrt{N+M}} + o\left(\frac{1}{\sqrt{N+M}}\right) \\ \rho(R,T_4), & если \quad p < \alpha, p < \frac{1}{\sqrt{N+M}} + o\left(\frac{1}{\sqrt{N+M}}\right) \\ \rho(R,T_3), & если \quad \alpha < 1-p, \alpha > \frac{1}{\sqrt{N+M}} + o\left(\frac{1}{\sqrt{N+M}}\right) \end{cases}$$

(58)

Conclusion

From (58) it follows that

$$\sup_{\substack{0\le\alpha,p\le 1\\ \sigma<\infty}} \min_{i=2,3,4,6,7} \rho(R,T_i) = (N+M)^{-\frac{1}{2}} + o\left((N+M)^{-\frac{1}{2}}\right) . \qquad (59)$$

It follows from (39) that the maximum values of $\rho(R,T_i)$, $i = 2,3,4,6,7$ α and p in these areas is reached at the point $A(2\lambda_2\alpha_1, 1-p_1)$:

$$2\lambda_2\alpha_1 = 1 - p_1 = \left(\frac{\lambda_1}{\lambda_2\sqrt{2\lambda_2}}\right)^{\frac{1}{2}} (N+M)^{-\frac{1}{4}} + o\left((N+M)^{-\frac{1}{4}}\right)$$

This is the maximum value of $\frac{\sqrt[4]{\lambda_1^2\lambda_2}}{\sqrt[4]{2}}(N+M)^{-\frac{1}{4}} + o\left((N+M)^{-\frac{1}{4}}\right)$

(60)

And for $i = 1,2,5$ Maximum values $\rho(R,T_i)$ reach at the point of

$$\alpha = p = \sqrt{\frac{\lambda_1}{\lambda_2}}(N+M)^{-\frac{1}{4}} + o\left((N+M)^{-\frac{1}{4}}\right).$$

This is the maximum value of $\sqrt{\lambda_1\lambda_2}(N+M)^{-\frac{1}{4}} + o\left((N+M)^{-\frac{1}{4}}\right)$ (61)

Now let's consider $\rho(R,T_i)$, $i = 1,3,4$. They reach their maximum values at the point of $A(1-\alpha_2, p_2)$:

$$1-\alpha_2 = p_2 = \sqrt{\frac{\lambda_1}{\lambda_2}}(N+M)^{-\frac{1}{4}} + o\left((N+M)^{-\frac{1}{4}}\right).$$

This is the maximum value of $\sqrt{\lambda_1\lambda_2}(N+M)^{-\frac{1}{4}} + o\left((N+M)^{-\frac{1}{4}}\right).$ (62)

From (60)-(62) follows the validity of theorem 1.

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