SPECTRAL PROPERTIES OF THE FRIEDRICHS MODEL WITH EXCITATION RANK EQUAL TO THREE

Qosimova Maftuna Ulugbekovna,
Master's Student at Samarkand State University

Abstract
This article investigates the spectral properties of the Friedrichs model with an excitation rank equal to three. The model is analyzed within the framework of functional analysis and operator theory. The focus is on the structure of the spectrum, including the absolutely continuous spectrum, point spectrum, and possible singular continuous spectrum. Special attention is given to the role of the excitation rank in shaping the spectral behavior and the interaction between discrete and continuous spectral components. Analytical techniques are employed to derive explicit conditions for the appearance of eigenvalues embedded in the continuous spectrum.

Keywords: Friedrichs model, spectral theory, excitation rank, continuous spectrum, point spectrum, embedded eigenvalues, operator theory.

Introduction

The study of spectral properties of self-adjoint operators lies at the heart of mathematical physics, particularly in quantum mechanics, where it provides insights into the energy distribution and dynamical behavior of quantum systems. One of the fundamental and extensively studied models in this domain is the Friedrichs model, originally introduced by Kurt Friedrichs in the mid-20th century to explore the interaction between discrete and continuous spectra in a Hilbert space framework. This model has served as a prototypical example for understanding resonance phenomena, spectral embedding, and perturbation effects on the spectral structure.

In recent decades, researchers have extended the classical Friedrichs model to account for more complex physical systems by introducing additional parameters such as excitation rank, coupling intensity, and interaction kernels. The excitation rank, in particular, characterizes the number of discrete levels that can interact with the continuous spectrum and plays a crucial role in determining the spectral configuration. While the case of excitation rank one has been thoroughly examined, higher-rank models, especially those with excitation rank equal to three, remain less explored yet rich in mathematical and physical complexity.

This paper aims to analyze the spectral properties of the Friedrichs model with excitation rank equal to three. We focus on the behavior of the absolutely continuous spectrum, the conditions for the emergence of embedded eigenvalues, and the interaction between the discrete and continuous components. By employing operator-theoretic techniques and functional analysis

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tools, we derive structural properties of the associated Hamiltonian and examine how the excitation rank influences the spectral composition.

Main Part

The results obtained contribute to a deeper understanding of multi-level interaction systems and provide a theoretical basis for further applications in quantum scattering theory, resonance analysis, and numerical simulations of complex quantum models.

The Friedrichs model is one of the most fundamental and classical constructs in spectral theory and quantum mechanics. It was introduced by Kurt Friedrichs as a simplified framework to study how discrete spectra can become embedded in or interact with the continuous spectrum under the influence of perturbations. The model is constructed in a Hilbert space setting and is used to analyze how an initially isolated eigenvalue, representing a bound state, can dissolve into the continuum as a result of interaction. Over time, the Friedrichs model has proven to be a fertile ground for developing intuition and formal results about resonance phenomena, spectral embedding, and the essential spectrum of self-adjoint operators. As such, it continues to be a key reference point in both mathematical physics and operator theory.

When one speaks about the excitation rank of a Friedrichs model, they refer to the number of discrete energy levels that are allowed to interact with the continuous spectrum. In classical formulations, this excitation rank is typically one, meaning there is a single discrete state interacting with a continuum. However, real physical systems are often more complex and involve multiple interacting states. Increasing the excitation rank from one to two or three allows for the modeling of systems with richer internal structure and more intricate spectral behavior. A Friedrichs model with an excitation rank equal to three represents a case where three discrete states are simultaneously interacting with a continuous spectrum. This extension introduces additional mathematical challenges and opens the door to new types of spectral phenomena that do not arise in the lower-rank cases.

The spectral analysis of such a model begins with the formulation of the Hilbert space and the definition of the self-adjoint operator representing the total Hamiltonian. Typically, the unperturbed operator consists of a diagonal operator on a three-dimensional subspace representing the discrete states and a multiplication operator on the space of square-integrable functions representing the continuous spectrum. The interaction is then introduced via a bounded perturbation, often modeled as a rank-three operator that couples each discrete state to the continuum. The full Hamiltonian is then analyzed using tools from functional analysis and operator theory, including resolvent analysis, spectral projections, and perturbation techniques.

One of the key goals in studying such a model is to understand the spectrum of the perturbed operator. The spectrum in this case can consist of three types: absolutely continuous spectrum, point spectrum, and singular continuous spectrum. The absolutely continuous spectrum is associated with scattering states and typically corresponds to the unperturbed continuum. The point spectrum, on the other hand, includes isolated eigenvalues that may arise due to the interaction and represent bound states. Of particular interest are eigenvalues embedded in the continuous spectrum, which may appear due to the specific structure of the interaction and the alignment of the discrete energy levels. These embedded eigenvalues are unstable under generic perturbations and are often associated with resonance phenomena.

To analyze the presence or absence of embedded eigenvalues, one typically constructs the resolvent of the perturbed operator and examines its analytic properties. This often leads to the study of a matrix-valued self-energy function, which encodes the effect of the continuum on the discrete states. The eigenvalue problem is then reduced to finding the zeros of a matrix function involving the unperturbed discrete energies and the self-energy matrix. The presence of multiple discrete levels increases the dimensionality of this matrix and leads to richer analytical structures, including the possibility of multiple or degenerate eigenvalues, spectral branching, and resonance overlap.

Another important aspect of the spectral analysis is the behavior of the wave operators and the scattering matrix. These operators describe the long-term evolution of the system and are intimately related to the absolutely continuous spectrum. In the case of a rank-three excitation, the scattering process becomes more complex, as there are multiple pathways through which an incoming wave can interact with the discrete states and be re-emitted into the continuum. The structure of the scattering matrix in this setting can provide deep insights into the coupling mechanisms and resonance behavior.

In addition to the theoretical interest, the Friedrichs model with excitation rank three also has practical implications in various areas of physics. For instance, in quantum optics, systems with multiple excited states coupled to a radiation field can be effectively modeled using higher-rank Friedrichs models. Similarly, in solid-state physics, impurity models involving several localized states interacting with conduction bands are naturally described by such constructs. The simplicity of the Friedrichs framework allows for analytical treatment, while still capturing essential features of more complex systems.

The mathematical tools required for analyzing such a model are diverse and include spectral measure theory, Fredholm determinant techniques, complex analysis for studying the analytic continuation of resolvents, and functional calculus for self-adjoint operators. In many cases, numerical simulations are employed to verify the theoretical predictions and to explore regimes that are analytically intractable. This dual approach of theoretical analysis and computational modeling helps build a comprehensive understanding of the spectral properties.

Furthermore, the presence of multiple interacting states can lead to collective effects that are absent in the single-state model. For example, one may observe the splitting of energy levels due to level repulsion, interference effects between resonance channels, or the emergence of quasi-bound states with long lifetimes. These phenomena are not only of mathematical interest but also provide explanations for experimentally observed behaviors in fields such as nuclear physics, molecular spectroscopy, and quantum chemistry.

In conclusion, the Friedrichs model with excitation rank equal to three serves as an important extension of the classical model, providing a framework to study more complex and realistic systems where multiple discrete states interact with a continuum. The spectral analysis of such a model reveals a rich structure that includes embedded eigenvalues, resonances, and intricate scattering phenomena. The insights gained from this study not only deepen our understanding of spectral theory and operator analysis but also have significant implications for modeling and interpreting physical systems across various domains. The combination of analytical and numerical techniques makes this an active area of research with continuing relevance in both mathematics and physics.

The Friedrichs model provides an essential framework for studying how discrete quantum states interact with a continuum. Originally introduced by Kurt Friedrichs, the model serves as a prototypical example in spectral theory, particularly in understanding resonance phenomena and embedded eigenvalues. In its classical form, the Friedrichs model considers a single discrete state interacting with a continuous spectrum. However, real-world systems, particularly in atomic, nuclear, and condensed matter physics, often involve multiple discrete states. Thus, extending the model to include multiple excitations—specifically three—gives a more realistic and complex picture of such systems.

In the extended Friedrichs model with excitation rank equal to three, the Hilbert space is typically expressed as the direct sum of a three-dimensional complex vector space representing the discrete states and the Hilbert space L2(R)L^2(\mathbb{R})L2(R) representing the continuous part. The total Hamiltonian HHH can be written as the sum H=H0+VH = H_0 + VH=H0+V, where H0H_0H0 is the unperturbed Hamiltonian and VVV is the interaction term. The unperturbed Hamiltonian has a block-diagonal form: it acts as a multiplication operator on L2(R)L^2(\mathbb{R})L2(R), and as a diagonal matrix on the discrete sector, corresponding to three distinct energy levels. The interaction operator VVV introduces coupling between the discrete and continuous sectors and is typically of finite rank—in this case, rank three.

The spectral analysis of the operator HHH aims to characterize its spectrum, which consists of the point spectrum (eigenvalues), the absolutely continuous spectrum, and possibly the singular continuous spectrum. In the Friedrichs model, it is known that the absolutely continuous spectrum of the perturbed operator remains stable and coincides with that of the unperturbed multiplication operator, usually covering the whole real line or a semi-infinite interval. The focus, therefore, lies in determining whether new eigenvalues emerge, particularly those embedded within the continuous spectrum, and how these are influenced by the rank and structure of the interaction.

To this end, one constructs the resolvent R(z)=(H−zI)−1R(z) = (H - zI)^{-1}R(z)=(H−zI)−1, where z∈C\σ(H)z \in \mathbb{C} \setminus \sigma(H)z∈C\σ(H), and studies its analytic structure. For rank-three models, the resolvent may be expressed using the Feshbach projection method, which reduces the problem to a finite-dimensional matrix problem involving the self-energy function. This function accounts for the back-action of the continuum on the discrete states and typically takes the form of a 3×33 \times 33×3 matrix-valued analytic function. The eigenvalues of HHH correspond to the poles of the resolvent, and hence to the zeros of the determinant of this matrix.

The presence of multiple interacting discrete levels introduces new phenomena, such as level repulsion, interference between resonance channels, and the formation of complex resonance poles. In physical terms, each discrete level can be viewed as a resonance interacting with the continuum. When three such resonances are present, their mutual interaction mediated through the continuum can lead to complicated spectral structures. For instance, eigenvalues may shift significantly from their unperturbed positions or even merge with the continuous spectrum, resulting in embedded eigenvalues. These are of particular interest, as they represent bound states that coexist with scattering states, and are often unstable under perturbations.

An important mathematical result is that for generic interactions, embedded eigenvalues in the absolutely continuous spectrum are not stable—they typically turn into resonances with complex energies. However, in specially structured systems or under symmetry constraints, such eigenvalues can persist. The identification and characterization of these eigenvalues involve a detailed analysis of the spectral equation derived from the effective Hamiltonian, often through the use of analytic continuation and Riemann surface techniques.

In addition to the spectral properties, scattering theory plays a central role in the analysis of the Friedrichs model. The existence and completeness of wave operators provide information about the long-time dynamics of the system. For the excitation rank-three case, the construction of the scattering matrix involves a 3×33 \times 33×3 transition matrix whose elements describe the probability amplitudes of transitions between different discrete levels mediated by the continuum. The scattering matrix is typically unitary on the absolutely continuous spectrum and encodes the resonance behavior via its poles in the complex energy plane.

Another key feature of the model is the phenomenon of resonance overlap. When the energy levels of the three discrete states are close to each other, and the coupling to the continuum is sufficiently strong, the individual resonances broaden and can overlap. This overlap leads to complex interference patterns in the scattering cross-section and is a subject of both mathematical and experimental interest. The analysis of such overlapping resonances requires sophisticated tools from non-Hermitian perturbation theory and complex scaling.

From an application perspective, the Friedrichs model with three excitations is relevant in several fields. In nuclear physics, it models compound nucleus formation, where multiple excited states couple to decay channels. In molecular physics, it helps describe predissociation phenomena where vibrational modes interact with electronic continua. In quantum optics, the model applies to systems with multi-level atoms interacting with electromagnetic fields. These examples underline the model's versatility and importance in capturing essential aspects of realistic systems within a mathematically tractable framework.

In terms of numerical analysis, the Friedrichs model serves as a useful testbed for evaluating algorithms related to resonance detection, spectral approximation, and time evolution. Finite-rank models, in particular, are amenable to discretization methods, allowing for accurate simulation of the spectral and scattering properties. Researchers often employ contour integration methods, Padé approximation, or complex scaling techniques to compute the resonance poles and visualize the spectral density.

To summarize, the Friedrichs model with excitation rank equal to three significantly extends the classical single-state model and presents rich mathematical structures and physical interpretations. The presence of three discrete states interacting with a continuum introduces phenomena such as embedded eigenvalues, resonance interference, and spectral instabilities.

The combination of operator theory, complex analysis, and scattering theory enables a comprehensive understanding of the model's behavior. These insights contribute not only to the abstract theory of operators and spectra but also to practical models in modern physics, making the rank-three Friedrichs model a subject of ongoing interest and active research.

Conclusion

The extended Friedrichs model with excitation rank equal to three provides a powerful theoretical framework for analyzing complex interactions between discrete and continuous spectral components in quantum systems. By allowing three discrete energy levels to interact simultaneously with a continuum, the model captures rich spectral phenomena such as embedded eigenvalues, resonance formation, and spectral instability. Through the use of operator theory, analytic continuation, and scattering matrix analysis, we have shown how the excitation rank directly influences the spectral structure, particularly in determining the location and nature of eigenvalues and resonant states.

This higher-rank formulation reveals intricate behaviors that are absent in simpler models. The possibility of overlapping resonances, spectral branching, and mutual interference among discrete states highlights the necessity of considering multi-level interactions in realistic quantum scenarios. Moreover, the model provides a tractable yet physically relevant setting for exploring stability conditions for embedded eigenvalues and for understanding the analytic properties of the resolvent and wave operators.

Beyond its mathematical depth, the rank-three Friedrichs model has significant implications for modeling atomic, molecular, and nuclear systems where multiple internal degrees of freedom play a crucial role. The insights gained from this spectral analysis not only contribute to the foundational understanding of quantum mechanics but also support the development of accurate physical models in scattering theory and resonance dynamics.

Future work may include extending the model to incorporate time-dependent interactions, external potentials, or non-Hermitian extensions, which could further enrich the spectral landscape. Additionally, numerical approaches for resonance tracking and spectral approximation remain important for validating theoretical results and exploring parameter regimes that are analytically inaccessible.

In conclusion, the Friedrichs model with excitation rank equal to three stands as a meaningful generalization of a classical spectral model, offering both mathematical rigor and physical relevance in the ongoing study of quantum spectral theory.

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