Volume 2, Issue 9, September 2024 **ISSN (E):** 2938-3757

THE FINDING PROBLEM FOR A NON-HOMOGENEOUS EQUATION WITH A MIXED-TYPE PRIVATE PRODUCT WITH TWO GENERATION LINES

Sharipov J. I.
Mirzo Ulugbek National University of Uzbekistan
jasurbeksharipov55555@gmail.com

Abstract
This work considers the boundary value problem for a heterogeneous mixed-type partial differential equation with two degeneracy lines, obtains a representation of the solution, obtains an a priori estimate of the solution, and obtains theorems proving the uniqueness and conditional stability of the solution in the set of correctness.

Keywords: incorrect problem, mixed type, a priori estimate, set of correctness, uniqueness, conditional stability.

КРАЕВАЯ ЗАДАЧА ДЛЯ НЕОДНОРОДНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ СМЕШАННОГО ТИПА С ДВУМЯ ЛИНИЯМИ ВЫРОЖДЕНИЯ

Шарипов Ж. И.
Национальный университет Узбекистана имени Мирза Улугбека
jasurbeksharipov55555@gmail.com

Аннотация
В данной работе рассматривается краевая задача для неоднородного уравнения в частных производных смешанного типа с двумя линиями вырождения, получено представление решения, выведена априорная оценка решения, получены теоремы, доказывающие единственность и условную устойчивость на множестве корректности решения.

Ключевые слова: некорректная задача, смешанного типа**,** априорная оценка, множество корректности, единственность, условная устойчивость.

Introduction

The work is dedicated to the investigated ill-posed boundary value problem for two degenerate lines of heterogeneous mixed-type equations.
The correct boundary value problems for such types of equations were considered in the works of A.M. Nakhushev, M.M. Zaynulabidov, V.F. Volkodovov, V.V. Azovsky, O.I. Marichev,

Web of Technology: Multidimensional Research Journal
webofjournals.com/index.php/4

A.M. Ezhov, N.I. Polivanov, He Kan Cher, S.I. Makarov, S.S. Isamuhamedov, Zh. Oramov, M.S. Salahitdinov with students, K.B. Repinov, A.A. Sab

In the sense of J. Adamar, incorrect problems were investigated in the works of E.M. Landis, S.G. Krein, M.M. Lavrentiev [1], H.A. Levine, I.E. Yegorov and V.E. Fedorov, A.I. Kozhanov, S.G. Pyatkov, A.L. Buchgeim, K.S. Fayazov [2], M.Kh. Alaminov, I.O. Khajiev [3], Ya.K. Hudayberganov [4],

Let $\Omega=\Omega_1\times\Omega_2\times Q$, Where $\Omega_1=\{-1<x<1,\ x\neq 0\}$, $\Omega_2=\{-1<y<1,\ y\neq 0\}$, $Q=\{0<t<T,\ T<\infty\}$.

Problem. It is required to find a function $u(x,y,t)$ related in the area Ω with equation

$$u_{tt}(x,y,t)+sgn(x)u_{xx}(x,y,t)+sgn(y)u_{yy}(x,y,t)+Au(x,y,t)=f(x,y,t), \quad (1)$$

and satisfying the conditions:

initial

$$\left.\frac{\partial^i u(x,y,t)}{\partial t^i}\right|_{t=0}=\varphi_i(x,y),\ (x,y)\in[-1;1]^2, \quad (2)$$

borderline

$$\begin{gathered} u_x(x,y,t)\big|_{\partial\Omega_1}=0,(y,t)\in[-1;1]\times\bar{Q}, \\ u_y(x,y,t)\big|_{\partial\Omega_2}=0,(x,t)\in[-1;1]\times\bar{Q}, \end{gathered} \quad (3)$$

and gluing conditions

$$\begin{gathered} \left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=-0}=(-1)^i\left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=+0},\ (y,t)\in[-1;1]\times\bar{Q}, \\ \left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=-0}=(-1)^i\left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=+0},\ (x,t)\in[-1;1]\times\bar{Q}, \end{gathered} \quad (4)$$

where, A is some constant, $(i=\overline{0,1})$, $\varphi_i(x,y)$, $f_i(x,y,t)$ is a given sufficiently smooth function, and $\varphi_{ix}(x,y)\big|_{\partial\Omega_1}=0$, $\varphi_{iy}(x,y)\big|_{\partial\Omega_2}=0$.

Spectral problem

Find such values λ at which the next task

$$sgn(x)\vartheta_{xx}(x,y)+sgn(y)\vartheta_{yy}(x,y)+\lambda\vartheta(x,y)=0,(x,y)\in(-1;1)^2,x,y\neq 0, \quad (5)$$

$$\begin{gathered} \vartheta_x(x,y)\big|_{x=\pm1}=0,\ \ y\in[-1;1],\ \vartheta_y(x,y)\big|_{y=\pm1}=0,\ \ x\in[-1;1] \\ \left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=-0}=(-1)^i\left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=+0},\ y\in[-1;1], \\ \left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=-0}=(-1)^i\left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=+0},\ x\in[-1;1],\ \left(i=\overline{0,1}\right), \end{gathered} \quad (6)$$

the solution to problem (5), (6) using the Fourier method, assuming

Web of Technology: Multidimensional Research Journal
webofjournals.com/index.php/4

$$\vartheta(x,y)=X(x)Y(y). \quad (7)$$

From conditions (6) we obtain:

$$\begin{aligned} &X'(-1)=X'(+1)=0,\\ &X(-0)=X(+0), \\ &X'(-0)=-X'(+0), \end{aligned} \quad (8)$$

$$\begin{aligned} &Y'(-1)=Y'(+1)=0,\\ &Y(-0)=Y(+0), \\ &Y'(-0)=-Y'(+0). \end{aligned} \quad (9)$$

Let's find the second partial derivatives of the function (7):

$$\vartheta_{xx}(x,y)=X''(x)Y(y), \quad \vartheta_{yy}(x,y)=X(x)Y''(y).$$

Substituting into (5) and separating the variables, we obtain:

$$\frac{sgn(x)X''(x)}{X(x)}+\frac{sgn(y)Y''(y)}{Y(y)}=-\lambda.$$

Further, since $\frac{sgn(x)X''(x)}{X(x)}$ does not depend on y, but $\frac{sgn(y)Y''(y)}{Y(y)}$ on x.

Thus, we have

$$\frac{sgn(x)X''(x)}{X(x)}=-\lambda_1, \quad \frac{sgn(y)Y''(y)}{Y(y)}=-\lambda_2.$$

As a result, to find the functions $X(x)$, $Y(y)$ we obtain the equations:

$$sgn(x)X''(x)=-\lambda_1 X(x), \quad (9)$$

$$sgn(y)Y''(y)=-\lambda_2 Y(y). \quad (10)$$

Let us consider equations (9), (10) with the corresponding conditions (8), (9)

$$\begin{aligned} &sgn(x)X''(x)=-\lambda_1 X(x),\\ &X'(-1)=X'(+1)=0,\\ &X(-0)=X(+0), \\ &X'(-0)=-X'(+0), \end{aligned} \quad (11)$$

$$\begin{aligned} &sgn(y)Y''(y)=-\lambda_2 Y(y),\\ &Y'(-1)=Y'(+1)=0,\\ &Y(-0)=Y(+0), \\ &Y'(-0)=-Y'(+0). \end{aligned} \quad (1\,2)$$

Thus, the solutions to problems (11) and (1 2) have the form:

at $\lambda_1>0,\ \lambda_2>0$,

$$X_k^{(1)}(x)=\begin{cases}\cos\mu_k(x-1)/\cos\mu_k, & 0<x<1,\\ ch\mu_k(x+1)/ch\mu_k, & -1<x<0,\end{cases} \quad k\in N,$$

$$Y_l^{(1)}(y)=\begin{cases}\cos\sigma_l(y-1)/\cos\sigma_l, & 0<y<1,\\ ch\sigma_l(y+1)/ch\sigma_l, & -1<y<0,\end{cases} \quad l\in N,$$

at $\lambda_1<0, \lambda_2<0$,

Web of Technology: Multidimensional Research Journal
webofjournals.com/index.php/4

$$X_k^{(2)}(x)=\begin{cases} ch\mu_k(x-1)/ch\mu_k, 0<x<1, \\ \cos\mu_k(x+1)/\cos\mu_k, -1<x<0, \end{cases} \quad k\in N,$$

$$Y_l^{(2)}(y)=\begin{cases} ch\sigma_l(y-1)/ch\sigma_l\ ,0<y<1, \\ \cos\sigma_l(y+1)/\cos\sigma_l\ ,-1<y<0, \end{cases} \quad l\in N.$$

at $\lambda_1=0, \lambda_2=0,$

$$X_0(x)=\begin{cases} 1/\sqrt{2}, & 0\le x\le 1, \\ 1/\sqrt{2}, & -1\le x\le 0, \end{cases}$$

$$Y_0(y)=\begin{cases} 1/\sqrt{2}, & 0\le y\le 1, \\ 1/\sqrt{2}, & -1\le y\le 0, \end{cases}$$

Where $\lambda_{1k}=\mu_k^2\ge 0,\ \lambda_{1k}=-\mu_k^2\le 0,\ \lambda_{2l}=\sigma_l^2\ge 0,\ \lambda_{2l}=-\sigma_l^2\le 0.$

Eigenvalues

$$\lambda_{k,l}^{(1)}=\mu_k^2+\sigma_l^2, \qquad \lambda_{k,l}^{(2)}=\mu_k^2-\sigma_l^2,$$
$$\lambda_{k,l}^{(4)}=-\mu_k^2+\sigma_l^2, \qquad \lambda_{k,l}^{(5)}=-\mu_k^2-\sigma_l^2,$$

correspond to their own functions

$$\vartheta_{k,l}^{(1)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(1)}(y), \quad \vartheta_{k,l}^{(2)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(2)}(y),$$
$$\vartheta_{k,l}^{(3)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(1)}(y), \qquad \vartheta_{k,l}^{(4)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(2)}(y),$$

Where $k,l=0,1,2,\ldots.$

In both cases, μ_k,σ_l – are non-negative roots of the transcendental equation $tg\alpha=-th\alpha.$ Let $\|u\|^2=(u,u)$, Where $(u,\upsilon)=\int_{-1}^{1}\int_{-1}^{1}u\upsilon dxdy$ scalar product . Besides

$$\left(\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(n)}(x,y)\right)=0, m\ne n, (m,n=\overline{1,4}),$$

$$\left(\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases} 1, & k=i\wedge l=j, \\ 0, & k\ne i\wedge l\ne j, \end{cases} \quad (m=\overline{1,4}),$$

Where $k,l,i,j\in N.$

Let

$$\|u(x,y,t)\|^2=\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(1)}(x,y)\right)\right|^2+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(2)}(x,y)\right)\right|^2$$
$$+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k}^{(3)}(x,y)\right)\right|^2+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(4)}(x,y)\right)\right|^2 \tag{13}$$

According to the Hilbert-Schmidt theorem [6], the eigenfunctions of problem (5)– (6) form a Riesz basis in $L_2(-1;1)^2$.

A priori estimate

Lemma 1. For any solution to problem (1)-(4) $t\in Q$ the inequality holds

Web of Technology: Multidimensional Research Journal
webofjournals.com/index.php/4

$$\int_0^t \|u(x,y,\tau)\|_0^2 d\tau \le 4q(t)\left(T\|u(x,y,0)\|_0^2+\alpha\right)^{1-p(t)}\left(\int_0^T \|u(x,y,t)\|_0^2 dt+\alpha\right)^{p(t)}, \qquad (14)$$

Where

$$\alpha=(2T^2+1)\int_0^T \|f(x,y,t)\|_0^2\,dt+2T\|\varphi_0\|_1^2+\|\varphi_0\|_0^2+(2T+1)\|\varphi_1\|_0^2,$$

$$p(t)=\frac{1-e^{-2t}}{1-e^{-2T}},\; q(t)=\exp\left(\frac{2T+1}{2}\frac{\left(1-e^{-2t}\right)T-\left(1-e^{-2T}\right)t}{1-e^{-2T}}\right).$$

The proof of this lemma can be found in [5] .
correctness M sets as follows

$$M=\left\{u(x,y,t):\int_0^T \|u(x,y,t)\|^2 dt \le m^2, m<\infty\right\}.$$

Uniqueness and conditional stability

Theorem 1. If the solution to the problem (1) - (4) exists and $u(x,y,t)\in M$, then the solution to problem (1) - (4) unique.

Proof . Let $u_1(x,y,t)$ and $u_2(x,y,t)$ be solutions to problems (1) - (4) with the same data. Then $u(x,y,t)=u_1(x,y,t)-u_2(x,y,t)$ will be the solution of problems (1) - (4) with a homogeneous equation and zero data. The function $u(x,y,t)$ is the solution of the corresponding homogeneous equation with zero data. This equation satisfies the conditions of Lemma 1, i.e. inequality (14) is true. From inequality (14) it follows $\|u(x,y,t)\|_0=0$. Consequently, for arbitrary $(x,y,t)\in\Omega, u(x,y,t)\equiv 0$ or $u_1(x,y,t)\equiv u_2(x,y,t)$. Theorem 1 is proven.
Let be $u(x,y,t)$ the solution of problem (1) - (4) with exact data, and $u_\varepsilon(x,y,t)$ let be the solution of problem (1) - (4) with approximate data.

Theorem 2 Let $u(x,y,t), u_\varepsilon(x,y,t)\in M$ And $\|\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)\|_1\le\varepsilon$, $\|\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y)\|_0\le\varepsilon$, $\|f(x,y,t)-f_\varepsilon(x,y,t)\|_0\le\varepsilon$. Then $U(x,y,t)=u(x,y,t)-u_\varepsilon(x,y,t)$ the inequality is true for the function

$$\int_0^t \|U(x,y,\tau)\|_0^2 d\tau \le 4q(t)\left\{T\varepsilon^2+\alpha_\varepsilon\right\}^{1-p(t)}\left\{4m^2+\alpha_\varepsilon\right\}^{p(t)},$$

for everyone $t\in Q$, Where $\alpha_\varepsilon=\varepsilon^2\left(2T^3+5T+2\right)$.

Web of Technology: Multidimensional Research Journal
webofjournals.com/index.php/4

Proofs of . Let function $U(x,y,t)$ is the solution to the corresponding problem (1) - (4), and $U(x,y,0)=\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)$, $U_t(x,y,0)=\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y)$. The function $U(x,y,t)$ satisfies the conditions of Lemma 1 and $\int_0^T \|U(x,y,t)\|_0^2 dt \le 4m^2$. Then for function $U(x,y,t)$ the following estimate is correct

$$\int_0^t \|U(x,y,\tau)\|_0^2 d\tau \le 4q(t)\left\{T\varepsilon^2+\alpha_\varepsilon\right\}^{1-p(t)}\left\{4m^2+\alpha_\varepsilon\right\}^{p(t)}.$$

Theorem 2 is proven .

REFERENCES

1. Lavrentiev M. M., Saveliev L. Ya. Operator Theory and Ill-Posed Problems. 2nd ed. Novosibirsk: Publishing House of the Institute of Mathematics, 2010, 941 p.
2. Fayazov K.S., Khazhiev I.O. Incorrect initial-boundary value problem for a system of parabolic equations with changing time direction. Computational technologies, Vol. 22, No. 3, 2017. P. 103-115.
3. Khazhiev I.O. Conditional stability of a boundary value problem for a system of high-order mixed-type equations. Scientific Bulletin of SSU , No. 1, 2022. P. 60-67.
4. Fayazov K. S., Khudaiberganov Ya. K., Ill-posed boundary value problem for a system of equations of mixed type with two degeneracy lines // Siberian Electronic Mathematical News. 2020. Vol. 17. pp. 647–660.
5. Kolmogorov A.N., Fomin S.V. Elements of the theory of functions and functional analysis. Moscow: Nauka . 1981. 544 p.

Web of Technology: Multidimensional Research Journal
webofjournals.com/index.php/4